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A New System of Consumer Demand Equations*

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ABSTRACT

This paper commences from a new indirect utility function and derives the corresponding system of equations, relating commodity demands to prices and income, that satisfies the constraints imposed by utility maximisation (aggregation, homogeneity, Slutsky symmetry and negativity). As the famous linear expenditure system (LES) is a special case of this new system, it is named the generalised Stone-Geary system (GSGS) and it incorporates a generalisation of the 'subsistance' income concept to one of 'committed' income. However, the GSGS is not subject to the well known limitations of the LES and it can model a reasonably representative range of consumer behaviour. It is also relatively parsimonious in parameters involving just 3n – 1, where n is the number of commodities. The new system has greater ranges of theoretical validity than various systems derived from 'flexible' functional forms. As with the LES, simple conditions on the parameters guarantee the validity of the system for all variable values except, perhaps, at low incomes.

I INTRODUCTION

A long standing approach to examining how consumers react to price and income changes estimates a set of demand equations for the main commodities and bases deductions on coefficient values. At its simplest, economic demand theory assumes consumers choose to allocate their limited spending power to purchases of goods to maximise their own satisfaction. This assumption of rational economic behaviour imposes substantial constraints (aggregation, homogeneity, Slutsky symmetry and negativity) on the specification of a system of equations. Ideally, of course, the system ought also be capable of representing the full range of observed consumer behaviour, for example, as regards substitutability and complementarity of various goods. Finally, data are often fairly limited, so models ought not to be too prolific in unknown parameters.

Compliance with optimisation theory can be assured by maximising a valid (direct) utility function, subject to a budget constraint, or alternatively, appealing to duality theory and commencing from a cost or indirect utility function. In the latter case, let $U(\mathbf{p}, y)$ be the indirect utility function, where **p** is a vector of prices and y is income. For validity, $U(\mathbf{p}, y)$ should be homogeneous of degree zero in income and prices (**p**), non-decreasing in y, non-increasing in **p**, and convex or quasi-convex in **p**. Then the demand equations can be obtained from Roy's lemma

$$q_i = -\frac{\partial U}{\partial p} / \frac{\partial U}{\partial y} \ ,$$

where q_i is the quantity demanded. For example, it is well known that the simple utility function

$$U = \frac{y}{P} \ , \tag{1}$$

where P is a weighted geometric mean of prices, so that $\log P = \Sigma(\alpha_j \log p_j)$, with the non-negative α_j adding to unity over the n commodities, conforms to the validity conditions. It gives the demand equations

$$w_i = \frac{p_i q_i}{y} = \alpha_i \, , \tag{2}$$

with w_i the budget shares. However, these equations limit consumer responses to changes in prices or income to maintaining equal proportional spending on commodities, that is, they restrict price and income elasticities to unity. While such a consumption pattern might sometimes be plausible, it would be unreasonable to assume it always is. Economists would usually like much more flexibility in the sense of price and income elasticities being able to take arbitrary values.

This paper will present a new system of demand equations that complies with optimisation theory, is flexible enough to represent a wide range of consumer behaviour and is reasonably parsimonious in unknown coefficients. As will be seen, it contains the famous Stone–Geary linear expenditure system (LES) as a special case and so will be subsequently described as the generalised Stone–Geary system (GSGS). However, over the last quarter century many demand systems have been introduced that have stressed their full flexibilities as their strengths, so it is probably necessary to first review the problem of simultaneously achieving compliance with rational economic behaviour[1] while maintaining flexibility as regards permissible elasticity values.

[1] Some authors use the term ‘regularity’ for this compliance, but that can suggest it is commonplace.

II FLEXIBILITY AND RATIONAL ECONOMIC BEHAVIOUR

It is actually very difficult to find functional forms for utility that permit flexibility for elasticities without transgressing the validity conditions for at least some combinations of prices and income. To illustrate the problems that can arise, suppose P in (1), is taken as a more general index over the n commodities of the form

$$\log P = \sum_j a_j \log p_j + \sum_j \sum_k b_{jk} \log p_j \log p_k \,, \qquad (3)$$

where the property of being homogeneous of degree one is ensured by supposing

$$\sum_j a_j = 1, \quad \sum_j b_{jk} = 0 \text{ for all k and } \sum_k b_{jk} = 0 \text{ for all j.}$$

This will lead to more flexibility of price elasticities, (athough income elasticities are still unitary). Obviously, the number of coefficients requiring estimation is now $O(n^2)$, which may raise difficulties if data are limited, but there is a more fundamental problem. A valid price index must increase with any commodity price, given other prices constant, but the quadratic (in logs) function (3) can first increase and then decrease with some price, because the conditions on the b_{jk} clearly imply some must be negative. That in turn means that the 'utility function' could increase with price (and fail convexity). It is *possible* that all 'real world' prices occurring in the data, or of interest in inference, might lie well away from the values where these phenomena would occur, so that the utility function could be 'locally' valid.

This type of difficulty arises with many flexible functional forms. For example, the translog utility function of Christiansen, Jorgensen and Lau (1975) is

$$\sum_j a_j \log \frac{y}{p_j} + \sum_j \sum_k b_{jk} \log \frac{y}{p_j} \log \frac{y}{p_k} \,,$$

with $\Sigma a_j = 1$ and $b_{jk} = b_{kj}$. This permits non-unitary income elasticities as well as non-unitary price elasticities, but if it is flexible it should be able to model homothetic behaviour and this requires $\sum_j b_{jk} = 0$ and $\sum_k b_{jk} = 0$. So these constraints must be permissible. But the utility function is then $\log y - \log P$, with log P exactly of the form (3) with the same constraints on coefficients. These comments are not original, of course. Caves and Christiansen (1980), Lau (1986) and Deaton (1986) have remarked on the incompatibility of flexibility and 'global regularity'.

But some flexible demand systems follow from utility functions that strain credibility much further. For example, the almost ideal demand system (AIDS) of Deaton and Muellbauer (1980)

$$w_i = a_i + \sum_j c_{ij} \log p_j + b_i \log \frac{y}{P^*},$$

where P^* is a price index[2], has as utility function

$$U = \frac{\log y - \log P(\mathrm{p})}{B(\mathrm{p})},$$

with $\log B = b_0 + \sum b_j \log p_j$, where $\sum b_j = 0$ and with log P again of the form (3) with the same constraints on parameters. It is easy to show that as y increases, U *must increase* with prices corresponding to negative b_j and, of course, $\Sigma b_j = 0$ implies some negative b_j. But even a low income region is not safe for validity. If U is quasi-convex in prices the diagonal terms of the matrix of second derivatives must not be negative. But

$$\frac{\partial^2 U}{\partial p_i^2} = \frac{1}{p_i^2 B}\left\{ w_i(1+2b_i) - c_{ii} - b_i^2 \log \frac{y}{P} \right\}.$$

Obviously this becomes negative as y increases, but even for small y there can be problems as $b_i < -.5$ shows. Again, these comments are not new. Criticisms of the AIDS model have appeared in the literature, for example, by Cooper and McLaren (1992)[3] and by Conniffe (1993). Similar difficulties arise for some other demand systems, for example for the various 'rank 3' systems such as that of Ryan and Wales (1999)[4].

So although these flexible forms can fit arbitrary patterns of elasticities, it is at a price of very shaky utility optimisation foundations, as well as $O(n^2)$ parameters. An alternative is to make optimisation, or rational economic behaviour, paramount in the choice of utility function and, subject to that, keep elasticities as flexible as possible. However, this has usually not been as flexible as desired. The LES can serve as an example. It can be derived[5] from the simple modification to (1)

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[2] Strictly, $\log P^* = \log P = a_o + \sum_j a_j \log p_j + \frac{1}{2}\sum_j \sum_k c_{jk} \log p_j \log p_k$, but to retain linearity for estimation simplicity, this is often approximated by $\log P^* = \sum_j w_j \log p_j$.

[3] Cooper and McLaren proposed a modification to the AIDS model – the MAIDS model (employed for example by Boyle, 1996), valid to higher income than AIDS, although it still loses validity when income increaseses sufficiently.

[4] That has $U = -g/(y-f)-h$, with g, f and h specified functions of prices. The crucial constraint that leads to violation of theory as income increases is $\Sigma\alpha_j = 0$ in h $= \Sigma(\alpha_j \log p_j)$.

[5] Its various historical derivations have been described by Neary (1997).

$$U = \frac{y - \Sigma\gamma_j p_j}{P}$$

and translates (in Gorman's, 1975, terminology) equation (2) to

$$q_i = \gamma_i + \frac{\alpha_i}{p_i}(y - \Sigma\gamma_j p_j),$$

with q_i presumed ≥ γ_i.. Provided, as with (1), the α_j are positive and add to unity, validity conditions hold[6]. But, as is well known, the LES is rather inflexible for representing observed consumer behaviour and is probably only appropriate for a set of broadly defined commodities. Complementarity between pairs of goods is precluded, as are negative income elasticities (inferior goods) and Engel curves (relationships of expenditures to income at fixed prices) are linear. As regards the latter property, one of the few generally agreed findings in empirical studies (see, for example, Lau, 1986) is that they can be non-linear for some goods.

Other simple and parsimonious utility functions, for example, Houthakker's (1960) indirect addilog system (IAD) lead to different, but also unwelcome, inflexibilities in the corresponding demand equations. So there is scope for any new system of demand equations that complies with utility optimisation, is flexible enough to represent a broad range of consumer behaviour and is reasonably parsimonious in unknown coefficients. The system to be presented in subsequent sections of this paper is one such. Compliance with utility theory will be assured by commencing from a utility function and imposing constraints only on its parameters. Empirical applicability will be shown by examining income, own-price and cross-price elasticities and demonstrating them free of obvious limitations. Furthermore, the system is quite parsimonious in involving only 3n - 1 parameters, at least in its simplest form.

[6] It's true $y > \Sigma\gamma_j p_j$ is required and the q_i must be non-negative, but failure of validity at low incomes is not considered a practical difficulty.

III THE GSGS INDIRECT UTILITY FUNCTION

Consider the indirect utility function

$$U = \frac{y}{P}\left\{1 - \sum \gamma_j \left(\frac{p_j}{y}\right)^{\beta_j}\right\}, \tag{4}$$

where y is income, P is a weighted geometric mean of prices, so that $\log P = \Sigma(\alpha_j \log p_j)$ with the non-negative α_j adding to unity, and summations are over the n commodities. U is presumed positive, that is

$$y > \sum \gamma_j p_j^{\beta_j} y^{1-\beta_j} \ ,$$

which will hold for all β_j positive provided y is not small[7]. If a β_i is negative, the corresponding γ_i must also be negative, while if a β_i is zero, the corresponding γ_i must be <1[8].

The first condition for a valid utility function is that it is homogeneous of degree zero in income and prices (**p**) and this is obviously true. Since

$$\frac{\partial U}{\partial y} = \frac{U}{y} + \sum \gamma_j \beta_j \left(\frac{p_j}{y}\right)^{\beta_j} \tag{5}$$

and

$$\frac{\partial U}{\partial p_i} = -\frac{\alpha_i U}{p_i} - \frac{\gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta_i - 1}}{P}, \tag{6}$$

the conditions of being nondecreasing in y and nonincreasing in **p** hold provided each $\gamma_i \beta_i$ is positive.

Finally, convexity or quasi-convexity in **p** is required. From (6)

$$\frac{\partial^2 U}{\partial p_i^2} = \frac{\alpha_i U}{p_i^2} - \frac{\alpha_i}{p_i}\left[\frac{\partial U}{\partial p_i}\right] - \frac{\gamma_i \beta_i (\beta_i - 1)\left(\frac{p_i}{y}\right)^{\beta_i - 2}}{yP} + \frac{\gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta_i - 1} \frac{\partial P}{\partial p_i}}{P^2}$$

$$= \frac{\alpha_i (1 + \alpha_i) U}{p_i^2} + \frac{2\alpha_i \gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta_i - 1}}{p_i P} - \frac{\gamma_i \beta_i (\beta_i - 1)\left(\frac{p_i}{y}\right)^{\beta_i - 1}}{p_i P}$$

[7] This is a generalisation of the condition on income required by the LES given in note 6.

$$= \left[\frac{\alpha_i^2 U}{p_i^2} + \frac{2\alpha_i\gamma_i\beta_i\left(\frac{p_i}{y}\right)^{\beta_i - 1}}{Pp_i}\right] + \left[\frac{\alpha_i U}{p_i^2} - \frac{\gamma_i\beta_i(\beta_i - 1)\left(\frac{p_i}{y}\right)^{\beta_i - 1}}{Pp_i}\right] =$$

$$\left[\frac{\alpha_i\sqrt{U}}{p_i} + \frac{\gamma_i\beta_i\left(\frac{p_i}{y}\right)^{\beta_i - 1}}{P\sqrt{U}}\right]^2 + \left[\frac{\alpha_i U}{p_i^2} - \frac{\gamma_i\beta_i(\beta_i - 1)\left(\frac{p_i}{y}\right)^{\beta_i - 1}}{Pp_i}\right] - \frac{\gamma_i^2\beta_i^2\left(\frac{p_i}{y}\right)^{2\beta_i - 2}}{P^2 U}. \qquad (7)$$

Also

$$\frac{\partial^2 U}{\partial p_i \partial p_k} = \frac{\alpha_i\alpha_k U}{p_i p_k} + \frac{\alpha_i\gamma_k\beta_k}{p_i P}\left(\frac{p_k}{y}\right)^{\beta_k - 1} + \frac{\alpha_k\gamma_i\beta_i}{p_k P}\left(\frac{p_i}{y}\right)^{\beta_i - 1} =$$

$$\left[\frac{\alpha_i\sqrt{U}}{p_i} + \frac{\gamma_i\beta_i}{P\sqrt{U}}\left(\frac{p_i}{y}\right)^{\beta_i - 1}\right]\left[\frac{\alpha_k\sqrt{U}}{p_k} + \frac{\gamma_k\beta_k}{P\sqrt{U}}\left(\frac{p_k}{y}\right)^{\beta_k - 1}\right] - \frac{\gamma_i\beta_i\gamma_k\beta_k}{P^2 U}\left(\frac{p_i}{y}\right)^{\beta_i - 1}\left(\frac{p_k}{y}\right)^{\beta_k - 1} \qquad (8)$$

It is evident from (7) and (8) that the Hessian matrix with respect to prices is composed of a nonnegative definite matrix GG', where G is the vector with ith term

$$\frac{\alpha_i\sqrt{U}}{p_i} + \frac{\gamma_i\beta_i}{P\sqrt{U}}\left(\frac{p_i}{y}\right)^{\beta_i - 1},$$

plus a diagonal matrix D with ith diagonal term equal to the second term of (7), minus a nonnegative definite matrix HH', where H is the vector with ith term

$$\frac{\gamma_i\beta_i}{P\sqrt{U}}\left(\frac{p_i}{y}\right)^{\beta_i - 1}.$$

While $-HH'$ is negative semi-definite, the addition to it of the diagonal matrix with ith term

$$\frac{2\gamma_i^2\beta_i^2\left(\frac{p_i}{y}\right)^{2\beta_i - 2}}{P^2 U}$$

gives a nonnegative definite matrix (diagonals positive and principal minors of higher order zero).

Subtracting the same matrix from D shows that U is convex with respect to prices provided all the terms

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[8] If several β_j are zero, the sum of the corresponding γ_j must be <1.

$$\frac{\alpha_i U}{p_i^2} - \frac{\gamma_i \beta_i (\beta_i - 1)\left(\frac{p_i}{y}\right)^{\beta_i - 1}}{P p_i} - \frac{2\gamma_i^2 \beta_i^2 \left(\frac{p_i}{y}\right)^{2\beta_i - 2}}{P^2 U}$$

or

$$\frac{\alpha_i P^2 U^2}{p_i^2} - \frac{\gamma_i \beta_i (\beta_i - 1)\left(\frac{p_i}{y}\right)^{\beta_i - 1} PU}{p_i} - 2\gamma_i^2 \beta_i^2 \left(\frac{p_i}{y}\right)^{2\beta_i - 2}$$

are positive for all i. Considering first the case of all β_j positive, the three components are of order y^2, $y^{2-\beta_i}$ and $y^{2-2\beta_i}$, so there is no difficulty provided α_i is positive. A zero β_i would leave only the first positive component. Note the second term is positive if β_i is < 1, so that a zero α_i may be compatible with validity in that situation. For at least one of the β_j negative, let β_s be the smallest (most negative). Then (since γ_i is negative) the three components are of order $y^{2-2\beta_s}$, $y^{2-\beta_i-\beta_s}$ and $y^{2-2\beta_i}$ and it is clear the critical term is i= s. For that term, the coefficient of $y^{2-2\beta_s} / p_s^{2-2\beta_s}$ is

$$\gamma_s^2 (\alpha_s - \beta_s - \beta_s^2),$$

which is certainly positive if β_s .> -1, however small α_s. (Extracting the term in $y^{1-\beta_s} / p_s^{1-\beta_s}$ from U will not leave a negative term if y is sizable because a positive term of order y, if β_s was the only Negative, or greater if there is another negative β will dominate.)

IV DEMAND EQUATIONS

Applying Roy's identity to the utility function (4) gives the GSGS demand equations

$$w_i = \frac{\alpha_i \left\{ 1 - \sum \gamma_j \left(\frac{p_j}{y} \right)^{\beta_j} \right\} + \gamma_i \beta_i \left(\frac{p_i}{y} \right)^{\beta i}}{1 - \sum \gamma_j \left(1 - \beta_j\right) \left(\frac{p_j}{y} \right)^{\beta_j}}, \tag{9}$$

in budget share form. The system of course satisfies the constraints of aggregation, homogeneity, symmetry and negativity, provided the conditions on parameters, specified in the previous section, are satisfied.

Taking all the β_i = 1 in (9) gives

$$w_i = \alpha_i \left(1 - \frac{\sum \gamma_j p_j}{y} \right) + \frac{\gamma_i p_i}{y}, \tag{10}$$

the linear expenditure system. So, there is a sense in which the demand system (9) may be seen as a generalisation of the linear expenditure system. This can be developed further. An attractive interpretation (when the γ's are non-negative, which they do not have to be) of the term $\Sigma\gamma_j p_j$ in (10) is as 'subsistence' income. When $(\Sigma\gamma_j p_j)/y$, is near 1, little more than essential quantities of each commodity are purchased and the demand system approximates $w_i = \gamma_i p_i / y$, or $q_i = \gamma_i$. So textbooks often (e.g. Deaton & Muellbauer, 1980, p.145) interpret (10) as giving a consumer's budget shares as a weighted average of a 'rich' person's and a 'poor' person's budget shares. The budget shares (9) can also be seen as weighted averages - now of those of a 'rich' person and of someone following the indirect addilog system. The quantity

$$y \sum \gamma_j \left(\frac{p_j}{y} \right)^{\beta_j} \tag{11}$$

could be considered a 'committed' income, although with wider interpretation than just subsistence income, since it can change with income. For β_i = 1 say, the ith component of (11) is $\gamma_i p_i$, so γ_i could again be taken as a minimum essential quantity purchased at price p_i irrespective of income. For β_k = 0 say, the kth component of (11) is $\gamma_k y$, so γ_k could be understood as a minimum proportion of income to be spent on commodity k irrespective of price and intermediate interpretations are possible for a β between zero and one. A β greater than one, where the commodity fades out of (11) as income increases, could be interpreted as a less drastic option than the exclusion at all incomes that setting the corresponding $\gamma = 0$ would imply. It is true the case of a negative β and its corresponding γ seems incompatible with this committed income notion, but that situation also arises with the LES when a γ is negative.

Assuming at least one β is less than unity, (11) increases to infinity as $y \to \infty$, with dominant term

$$\gamma_s p_s^{\beta_s} y^{1-\beta_s} ,$$

where β_s is the smallest of the β_i. If all β_i are < 1, (11) goes to zero with y with dominant term

$$\gamma_l p_l^{\beta_l} y^{1-\beta_l} ,$$

where β_l is the largest of the β_i. However, if one or more $\beta' s \geq 1$, y cannot go to zero. In the LES the condition $y > \Sigma\gamma_j p_j$ means the system is inapplicable at low y. Here (11) may go to zero with y, perhaps giving a somewhere wider validity to the GSGS. But it requires all $\beta' s < 1$ and in any case invalidity of the LES at low income has never been seen as important in the literature.

Whatever about the interpretation of (11) as committed income, it is clear that when income is little greater than (11), that is when

$$\sum \gamma_j \left(\frac{p_j}{y} \right)^{\beta_j}$$

approaches unity, the demand system (9) tends to

$$w_i = \frac{\gamma_i \beta_i \left(\frac{p_i}{y} \right)^{\beta i}}{\sum \gamma_j \beta_j \left(\frac{p_j}{y} \right)^{\beta_j}} , \qquad (12)$$

the IAD[9] demand system. Indeed (9) could be seen as a weighted average of a ‘rich’ person’s and an ‘IAD’ person’s budget shares.

[9] There are relations between IAD and GSGS validity conditions. The validity conditions of the IAD have been debated in the literature more than once, as the exchanges between Gamelatos (1973, 1974) and Somermayer (1974), and between Akin and Stewart (1979) and Murty (1982) testify.

The limiting form of (9) for large y is determined by the smallest β_i. By multiplying numerator and denominator of (9) by y to the power of β_s and letting $y \to \infty$, it turns out that for β_s negative

$$w_s = \frac{\alpha_s - \beta_s}{1-\beta_s} \quad \text{and} \quad w_i = \frac{\alpha_i}{1-\beta_s}, \qquad \text{for } i \neq s,$$

while for β_s non-negative, $w_i = \alpha_i$, for all i, which is the same limiting form as for the linear expenditure system. If all β_i are < 1, the largest, β_l, plays a corresponding role as $y \to 0$, with

$$w_l = \frac{\alpha_l - \beta_l}{1-\beta_l} \quad \text{and} \quad w_i = \frac{\alpha_i}{1-\beta_l}, \qquad \text{for } i \neq l,$$

and $w_i = \alpha_i$, for all i, depending on whether β_l is positive or negative. These limiting forms are quite different[10] to those of (12), where $w_s \to 1$ as $y \to \infty$ and $w_l \to 1$ as $y \to 0$ and all other budget shares to zero.[11]

[10] Except when all α_i but one are zero.

[11] This feature of the IAD might be plausible if goods could be reclassified extracting the most expensive components from the standard commodity classifications to form a super-luxury class (and a corresponding most basic-necessity class), but is untenable given conventional National Accounting and Budget Survey classifications.

V INCOME AND PRICE ELASTICITIES AND ENGEL CURVES

Following some tedious differentiation and tidying of terms, the income elasticity for the ith commodity can be shown to be

$$E_i = \frac{\alpha_i}{w_i} - \beta_i + \sum w_j \beta_j + \frac{V_1}{V_2}\left\{1 - \sum \alpha_j \beta_j - \frac{\alpha_i(1-\beta_i)}{w_i}\right\}, \tag{13}$$

where

$$V_1 = 1 - \sum \gamma_j \left(\frac{p_j}{y}\right)^{\beta_j} \quad \text{and} \quad V_2 = 1 - \sum \gamma_j (1-\beta_j)\left(\frac{p_j}{y}\right)^{\beta_j}.$$

The expression (13) permits goods to be luxuries or necessities at finite income, according as E_i is larger or smaller than unity. E_i could even be negative for some range of income (for a positive β_i and small α_i), so permitting a commodity to behave as an inferior good. But if α_i is non zero, E_i must eventually become positive as income increases since it was shown in the previous section that as $y \to \infty$ the budget shares tend to constants and therefore income elasticity to unity. This would be reasonable enough for a commodity like public transport, say, where expenditure on bus and train fares could decrease with rising income as private motor ownership increases and then increase again as more expensive public transport options, such as taxies, are availed of. Mathematically, if $\alpha_i = 0$, the limit of E_i as $y \to \infty$ is 1 - β_i, or 1 - $\beta_i + \beta_s$, depending on the sign of β_s, suggesting the good could be 'permanently inferior'. However, section III showed that $\alpha_i = 0$ could only be compatible with convexity if $\beta_i < 1$.

The own-price elasticities are

$$e_{ii} = -1 + \{\beta_i(1-w_i) + w_i - \alpha_i\}\left(1 - \frac{\alpha_i}{w_i}\frac{V_1}{V_2}\right),$$

which tends to –1 as $y \to \infty$, provided α_i is non-zero and to β_i - 1, if it is zero. The cross-price elasticities are

$$e_{ik} = \left(1 - \beta_k - \frac{\alpha_i}{w_i}\right)\left(w_k - \frac{V_1}{V_2}\alpha_K\right).$$

As $y \to \infty$ the e_{ik} evidently tend to zero. Note that e_{ik} need not equal e_{mk}. A limitation of the IAD is that the cross-price elasticities of all goods with respect to p_k are equal, a property arising (as pointed out by Samuelson, 1965) from the additivity of its indirect utility function, which implies various behavioural restrictions, such as the impossibility of luxuries having complements.

The compensated price effect

$$\frac{\partial q_i}{\partial p_k} + q_k \frac{\partial q_i}{\partial y}$$

is

$$\frac{q_i q_k}{y}\{1-\beta_i-\beta_k+\Sigma w_j\beta_j+\frac{V_1}{V_2}(1-\Sigma\alpha_j\beta_j)\}+\frac{V_1}{V_2}\{\frac{\alpha_i\alpha_k y}{p_i p_k}-\frac{q_i\alpha_k}{p_k}(1-\beta_k)-\frac{q_k\alpha_i}{p_i}(1-\beta_i)\}$$

and this can generally be negative or positive, allowing for complements as well as substitutes unlike the situation where all β_i are unity (giving the linear expenditure system), when the above reduces to

$$\left(y-\Sigma\gamma_j p_j\right)\frac{\alpha_i\alpha_k}{p_i p_k} \ ,$$

which is non-negative.

The Engel curves are the relationships between expenditure ($p_i q_i$) on commodities and income at constant prices. That these are not restricted to linearity, but can take a large variety of shapes, is probably already obvious from equation (9) and the income elasticity (13). Linearity will of course occur if all the β_i are unity. The limiting forms for the w_i described in section IV imply $p_i q_i \approx k\, y$, (with k a constant) for very large y (and near y = 0, provided all $\beta_i < 1$), so Engel curves will then approximate linearity, but before that, shapes can vary greatly. It is probably unnecessary to illustrate this in detail, especially since Somermayer and Langhout (1972) devoted much effort to demonstrating the great range of Engel curves arising from the IAD, which, as already mentioned, is close to a special case of the GSGS.

VI LIMITS TO FLEXIBILITY AND GENERALISATIONS

The previous section has demonstrated that the GSGS is considerably more flexible than the LES or IAD models. But are there limits to its flexibility? In discussing this theme, it is useful to consider the class of demand functions Pollak (1972) described as exhibiting "generalised separability", although (9) does not actually belong to the class. These are of the form

$$q_i = f_i(\frac{p_i}{y}, V),$$

where V is a homogenous function of all prices and income and because all prices, except p_i, take effect through V there are implied symmetries in how demands for commodities are affected by prices of other goods. For example, for the IAD as given by (11), V is the denominator and so the ratio of two budget shares does not depend on the prices of other commodities. While this may not always be implausible, it is restrictive.

The GSGS demand equations may be written

$$w_i = \frac{1}{V_2}\left[\alpha_i V_1 + \gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta i}\right],$$

with V_1 and V_2 defined as in section V. Although prices, other than own price, take effect through V_1 and V_2, there is now scope (via the α_ι) for greater variation in how commodity demands are affected by prices. However, there are still restrictions in limiting other price effects to operate through V_1 and V_2. For example, if a subset of commodities had α's zero, then within this subset, ratios of budget shares would depend, as for the IAD, only on the corresponding pairs of prices. This is hardly a worrying restriction, but does show there are limits to the system's flexibility to represent all conceivable consumer behaviour. This is partly due to the fairly parsimonious parameterisation of the model, but also to the fact that the constraints implied by consumer demand theory are not at all trivial in the limitations they impose on functional forms.

Since the underlying motivation for this paper is that 'flexibility' is rather useless if it means a model is able to represent many kinds of consumer behaviour except that corresponding to rational economic optimisation, generalisations of the GSGS utility function or the demand equations have to be guarded. A device like taking P of the form (3), would endanger compliance with demand theory and, perhaps less importantly, greatly increase the number of parameters. A modest generalisation of P is feasible though. In (4), P was taken to be a weighted geometric mean of prices. This is not the only simple form of price index and a weighted arithmetic or harmonic mean of prices could also have been employed. In fact, by introducing an extra parameter it is possible to encompass all three forms of price index in the formula

$$P = \left[\Sigma\alpha_j p^{\theta}{}_j\right]^{\frac{1}{\theta}},$$

where θ = 1 gives a weighted arithmetic mean, θ = -1 gives a weighted harmonic mean and θ = 0 gives (by a limit argument) a weighted geometric mean. If all the $\beta_i = 1$, these amount to the variants of the LES that have been examined by Pollak (1971) and Gamelatsos[12] (1973). These variants only slightly increased the flexibility of the LES and the situation seems similar for the GSGS.

The device of 'translating' (Gorman, 1975) whereby income y in a utility function is replaced by $y - \Sigma\phi_j p_j$, altering demand equations from $q_i = q_i(\mathrm{p}, y)$ to

$$q_i = \phi_i + q_i(\mathrm{p}, y - \Sigma\phi_j p_j),$$

can be employed to produce the demand equations

$$w_i = \frac{\phi_i p_i}{y} + \frac{1}{V_2^*}\left[\alpha_i V_1^* + \gamma_i\beta_i\left(\frac{p_i}{y - \Sigma\phi_j p_j}\right)^{\beta i}\right](1 - \frac{\Sigma\phi_j p_j}{y}), \qquad (14)$$

where the V*' s follow from the V's by replacing y by $y - \Sigma\phi_j p_j$. Another n parameters have been introduced here, but not all that much extra flexibility can be obtained from translation parameters and there can be problems of over-parametrisation and identifiability that could cause difficulties when estimating models. For example, it is well known the LES can be derived from a homothetic system by translation, which would correspond to putting all the $\beta_i = 0$ in (14). But as shown earlier, (9) reduces to the LES when all the $\beta_i = 1$. So the sub-models of (14): all $\beta_i = 0$; and, all $\beta_i = 1$ with $\phi_i = 0$; are identical.

The form of the utility function (4) suggests that the GSGS can be considered one of a family deduced from utility functions generated from products of simpler utility functions. But products of such functions need not satisfy all validity conditions, so care is required. Varying the first utility, y/P, by use of other forms of price mean has already been mentioned and it is easy to verify that replacing the second (IAD type) utility by some simple utility

[12] Gamelatsos did not actually work from an indirect utility function, but from direct utility functions of the Bergson type, previously considered by Samuelson (1965). So did Brown and Heien (1972).

functions also achieves convexity of the product function[13]. But these simple utilities have fewer parameters than the IAD, so the final demand functions with just 2n – 1 parameters are less flexible than (9). However, choosing a second utility with more parameters than the IAD type and demonstrating the validity of the resultant product seems difficult.

[13] Deriving demand systems from suns or products of utility functions is the subject of a working paper by Conniffe (2002).

VII CONCLUDING REMARKS

Detailed illustration of estimation and testing of the GSGS model on Irish data will be deferred to another paper, but, in any event, will not require any new methodological innovations. Evidently, estimation is not computationally simple and requires non-linear methods, but in this regard it is no more complex than the translog, or the AIDS model in its strict non-linear form. Hypothesis testing issues will also arise, of course, especially as regards the conditions on parameters, such as non-negative α_j and positive $\gamma_j\beta_j$ required for rational economic optimising behaviour, since random variation could give an apparently wrong sign estimate that is not actually significantly different from zero. To test some hypotheses, for example homogeneity, it may be necessary to estimate models with extra parameters. Again however, the testing procedures will be much the same as for some existing demand systems.

But there is one topic that perhaps has data analysis implications and which deserves some discussion here, because it is interrelated with the functional form of the GSGS model. The demand equations (in expenditure form) do not aggregate over consumers except in the LES special case. That is, averaging individual expenditures on goods over a group of consumers of varying incomes (presumed facing the same prices and with the same equation parameters applicable) does not reproduce the same equations with income replaced by average income. The LES case aggregates because the equations are then linear. Requiring aggregation over consumers to hold is extremely restrictive on the choice of functional form of demand equations. If the data for analysis are records on individuals the property, or the lack of it, is irrelevant. But data are very often already aggregated over individuals to some degree. Some authors believe utility maximisation, and the consequent constraints on demand equations, pertain strictly to individuals and consequently consider they will apply at aggregate level only if the demand equations aggregate over consumers. Other authors feel it quite appropriate to conceptualise a hypothetical consumer corresponding to the aggregate data and to visualise this representative consumer as a utility maximiser. Some, indeed, have been sceptical of the value of applying utility theory to actual individuals at all, rather than to representative consumers. The arguments on this matter are beyond the scope of this paper, but for some economists, the range of applicability of the GSGS may depend on the views they hold on this topic.

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