GENERATING GLOBALLY REGULAR INDIRECT UTILITY FUNCTIONS

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Abstract

Despite their scarcity in the literature, an abundance of globally regular indirect utility functions, involving as many parameters as desired, exists. They are easily constructed as a function of simple homothetic component utilities.

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1. INTRODUCTION

Regularity means that an indirect utility function complies with the constraints implied by rational economic behaviour[1]. Regularity is global if it holds for all (positive) prices and incomes, given appropriate ranges for the values of parameters occurring in the function. This short paper is concerned with the generation of such globally regular functions.

2. COMBINING HOMOTHETIC COMPONRNTS

Consider a set of homothetic utility functions, each of the form

$$U_k(\mathbf{p}, y) = y^{\delta_k} / P_k, \tag{1}$$

where δ_k is positive and P_k is increasing in prices, homogeneous of degree δ_k in prices and is a concave function of prices with negative definite or semi-definite Hessian. Then the reciprocal of P_k is convex in prices, because

$$\frac{\partial^2 P_k^{-1}}{\partial p_i^2} = \frac{2}{P_k^3}\left(\frac{\partial P_k}{\partial p_i}\right)^2 - \frac{1}{P_k^2}\frac{\partial^2 P_k}{\partial p_i^2}$$

and

$$\frac{\partial^2 P_k^{-1}}{\partial p_i \partial p_j} = \frac{2}{P_k^3}\left(\frac{\partial P_k}{\partial p_i}\right)\left(\frac{\partial P_k}{\partial p_j}\right) - \frac{1}{P_k^2}\frac{\partial^2 P_k}{\partial p_i \partial p_j},$$

so the Hessian of P_k^{-1} is

$$\frac{2}{P_k^3}\left(\frac{\partial P_k}{\partial \mathbf{p}}\right)\left(\frac{\partial P_k}{\partial \mathbf{p}}\right)',$$

which is positive semi-definite, minus the Hessian of P_k, which also gives a nonnegative definite matrix. So $U_k(\mathbf{p}, y)$ is convex in **p,** non-decreasing in y, non-increasing in **p** and as well as homogeneous of degree zero in income y and prices. So it is globally regular. A few properties of U_k are worth examining. As is already obvious, U_k^{-1} is a concave function of prices. For positive $\lambda \le 1$, $\left(U_k^{-1}\right)^\lambda = U_k^{-\lambda}$, being an increasing concave function of a concave function, is concave in

[1] That is, a consumer maximises direct utility under a budget constraint. This implies the indirect utility function $U(\mathbf{p}, y)$ should be homogeneous of degree zero in income y and prices **p**, non-decreasing in y, non-increasing in **p**, and convex or quasi-convex in **p.** These constraints imply corresponding conditions (aggregation, homogeneity, Slutsky symmetry and negativity) on the demand equations.

prices. For $\rho \geq 1$, $(U_k)^\rho$ is an increasing convex function of a convex function and so is convex in prices. For $\rho < 1$, it is simple to verify that $(P_k)^\rho$ has a negative definite or semi-definite Hessian and therefore its reciprocal is convex in prices and so is $(U_k)^\rho = U_k^\rho = y^{\rho\delta_k} / P_k^\rho$. So $(U_k)^\rho$ is convex in prices for all positive ρ. Also, $\log U_k$ is convex in prices as it equals $\delta_k \log y - \log P$ and since $\log P$ is an increasing concave function of a concave function, it is concave and minus it is convex.

Now consider the function

$$U = \left[\sum \phi_k U_k^{-\lambda}\right]^{-\frac{1}{\lambda}}, \tag{2}$$

where the φ_k are non-negative and sum to unity[2]. Commence with positive $\lambda \leq 1$. $U_k^{-\lambda}$ is concave in prices and since sums of concave functions are concave, $\sum \phi_k U_k^{-\lambda}$ is concave. Then

$$\left[\sum \phi_k U_k^{-\lambda}\right]^{\frac{-1}{\lambda}}$$

is a decreasing function of a quasi-concave function and so is quasi-convex. Since

$$\frac{\partial U}{\partial y} = \frac{1}{y}\left[\sum \phi_k U_k^{-\lambda}\right]^{-\frac{1}{\lambda}-1} \sum \delta_k \phi_k U_k^{-\lambda}$$

and

$$\frac{\partial U}{\partial p_i} = -\left[\sum \phi_k U_k^{-\lambda}\right]^{-\frac{1}{\lambda}-1} \sum \phi_k U_k^{-\lambda} \frac{1}{P_k}\frac{\partial P_k}{\partial p_i}$$

U is increasing in income and decreasing in prices. It is obviously homogeneous of degree zero in income and prices since each U_k is. So U is globally regular for $\lambda \leq 1$.

Now take λ negative and put $-\lambda = \rho$. Then (2) becomes

$$U = \left[\sum \phi_k U_k^\rho\right]^{\frac{1}{\rho}}.$$

As already shown U_k^ρ is convex in prices, so $\sum \phi_k U_k^\rho$ is convex and since an increasing function of a convex function is quasi-convex, U is quasi-convex in prices. It is clear that U is again increasing in income, decreasing in prices and homogeneous of degree zero in income and prices. Finally, for

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[2] The function (2) is of the form of the reciprocal of a CES (constant elasticity of substitution) price index, although here it is an index of component utilities not prices.

$\rho = 0$, the familiar limiting argument as $\rho \to 0$ (for example, Diewert, 1993) gives

$$U = \prod U_k^{\phi_k} \quad \text{or} \quad \log U = \sum \phi_k \log U_k \, .$$

Since each $\log U_k$ is convex in prices, the sum is, and since antilog is an increasing convex function, U is. So (2) gives a globally regular utility function for all $\lambda \leq 1$.

For utilities of the form y / P_k, which are special cases of (1), the sums, products and harmonic means, which are special cases of (2), have been examined in Conniffe (2002). However, they are of limited interest as the resulting utilities must all be homothetic.

3. EXAMPLES

Consider

$$U_1 = \frac{\sqrt{y}}{\sum \alpha_j \sqrt{p_j}} \qquad \text{and} \qquad U_2 = \frac{y}{\sum\sum \xi_{ij} \sqrt{p_i}\sqrt{p_j}} \, .$$

Both are easily shown to be convex in prices, the former strictly so, provided the α_j and ξ_{ij} are positive and $\xi_{ij} = \xi_{ji}$. Take $\phi_i = 2/3, \phi_2 = 1/3$ and $\lambda = 1$, so that (2) is a weighted harmonic mean of U_1 and U_2. It is

$$3\left[2\sum \alpha_j \left(\frac{p_j}{y}\right)^{\frac{1}{2}} + \sum\sum \xi_{ij} \left(\frac{p_i}{y}\right)^{\frac{1}{2}} \left(\frac{p_j}{y}\right)^{\frac{1}{2}}\right]^{-1}, \tag{3}$$

Diewert's (1974) generalised Leontief utility function. Diewert proved its global regularity, but that is immediately evident from the derivation here. More importantly, (3) is just one of many possibilities, all globally regular. Taking $\lambda = -1$, with the same ϕ values as before, gives a weighted arithmetic mean of U_1 and U_2

$$\frac{2}{3}\left[\sum \alpha_j \left(\frac{p_j}{y}\right)^{\frac{1}{2}}\right]^{-1} + \frac{1}{3}\left[\sum\sum \xi_{ij} \left(\frac{p_i}{y}\right)^{\frac{1}{2}} \left(\frac{p_j}{y}\right)^{\frac{1}{2}}\right]^{-1} . \tag{4}$$

The demand systems following from (3) and (4) are not the same and show differences that are possibly important in practice. For example, the income elasticities of the generalised Leontief demand system are

$$E_i = \frac{1}{2} + \frac{\frac{\alpha_i}{w_i}\left(\frac{p_i}{y}\right)^{\frac{1}{2}} + \sum\sum \xi_{ij}\left(\frac{p_i}{y}\right)^{\frac{1}{2}}\left(\frac{p_j}{y}\right)^{\frac{1}{2}}}{2\left[\sum \alpha_j \left(\frac{p_j}{y}\right)^{\frac{1}{2}} + \sum\sum \xi_{ij}\left(\frac{p_i}{y}\right)^{\frac{1}{2}}\left(\frac{p_j}{y}\right)^{\frac{1}{2}}\right]},$$

which must be greater than a half. The income elasticities corresponding to (4) are

$$E_i = \frac{3}{2} - \frac{\frac{\alpha_i}{w_i}\left(\frac{p_i}{y}\right)^{\frac{1}{2}} + \left(\sum \alpha_j p_j^{\frac{1}{2}}\right)^2 / \sum\sum \xi_{ij} p_i^{\frac{1}{2}} p_j^{\frac{1}{2}}}{2\left[\sum \alpha_j \left(\frac{p_j}{y}\right)^{\frac{1}{2}} + \left(\sum \alpha_j p_j^{\frac{1}{2}}\right)^2 / \sum\sum \xi_{ij} p_i^{\frac{1}{2}} p_j^{\frac{1}{2}}\right]},$$

which must be less than 3/2, which might be more acceptable than constraining them to exceed a half. Regularity need not imply the capability to model a wide range of consumer behaviour. The generalised Leontief's deficiencies in this regard have been mentioned by Caves and Christensen (1980) and Diewert and Wales (1987) and (4) has its own inflexibilities. Hoewever, λ, or a ϕ, could be treated as an unknown parameter to gain more flexibility. The case of $\lambda = 0$, giving products of utilities is particularily inflexible since the resulting function is still homothetic.

Many other choices of U_1 and U_2 are obviously possible; for example, U_1 could be taken to be

$$\frac{y^{\delta_1}}{\prod p_j^{\beta_j}},$$

where $\Sigma\beta_j = \delta_1$. Using (2) to combine this with the previous U_2 with λ=1 and $\phi_1 = \phi_2 = 1/2$ gives

$$2\left[\prod\left(\frac{p_j}{y}\right)^{\beta_j} + \sum\sum \xi_{ij}\left(\frac{p_i}{y}\right)^{\frac{1}{2}}\left(\frac{p_j}{y}\right)^{\frac{1}{2}}\right]^{-1},$$

and the corresponding demand equations are

$$w_i = \frac{\beta_i \prod\left(\frac{p_j}{y}\right)^{\beta_j} + \left(\frac{p_i}{y}\right)^{\frac{1}{2}} \sum \xi_{ij}\left(\frac{p_j}{y}\right)^{\frac{1}{2}}}{\delta_1 \prod\left(\frac{p_j}{y}\right)^{\beta_j} + \sum\sum \xi_{ij}\left(\frac{p_i}{y}\right)^{\frac{1}{2}}\left(\frac{p_j}{y}\right)^{\frac{1}{2}}}.$$

Clearly, three or more component utility functions could be employed in (2) to produce even more heavily parameterised, but still globally regular, utility functions. While the flexibility of the resulting demand systems would be increased, especially if λ, or the ϕ were also parameterised, the data requirements for the estimation of so many parameters could be a serious practical difficulty.

4. DISCUSSION

This paper has shown how to generate highly parameterised globally regular utility functions from (2) using the globally regular components (1). Familiar and more parsimonious utility functions could also be seen as following from (2). Houthakker's (1960) indirect addilog function

$$\sum \gamma_j \left(\frac{y}{p_j} \right)^{\beta_j},$$

where γ_i and β_i have the same sign and β_i>-1, is globally regular and is obviously (2) applied to y/p_j to the power of β_j with $\lambda = -1$ and $\phi_j = \gamma_j$. The components employed in section 3 could be considered as resulting from application of (2) to still simpler components. For example, let $U_k = \sqrt{y/p_k}$, a utility function corresponding to expenditure of the whole budget on good k. Using (2) with $\lambda = 1$ and $\phi_j = \alpha_j$ gives the U_1 of section 3. $U_{ik} = y/\sqrt{p_i p_k}$, a utility function corresponding to expenditure of half the budget on good i and half on good k, is (2) applied to U_i and U_k with $\lambda = 0$ and $\phi_i = \phi_k = 1/2$. Then applying (2) to U_{ik} with $\lambda = 1$ and ξ for ϕ gives U_2 of section 3.

Of course, many utility functions have appeared in the literature that are not of the class generated by (2) from components of the form (1). But they are not globally regular.

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