Generalised Translation of Indirect Utility Functions

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Abstract
This paper considers the derivation of new demand systems from existing ones through replacing an indirect utility function $U(\mathbf{p}, y)$ by $U\{\mathbf{p}, y - y\Sigma\gamma_j(p_j / y)^{\beta_j}\}$, where **p** is a vector of prices and y is income. This is a generalisation of Gorman translation $U(\mathbf{p}, y - \Sigma\gamma_j p_j)$ and will be shown to be effective in terms of producing new demand systems with both good regularity and flexibility properties.

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I INTRODUCTION

Gorman (1975) introduced the "translation" device to incorporate extra parameters into utility functions and demand equations. If $U(\mathbf{p}, y)$ is the original indirect utility function, where **p** is a vector of prices and y is income, the translated utility function is

$$U(\mathbf{p}, y - \Sigma\gamma_j p_j), \qquad (1)$$

where the γ_j are the subsistence quantities, y is assumed $> \Sigma\gamma_j p_j$ and summation is over n commodities. Gorman showed that if $q_i(\mathbf{p}, y)$ were the original demand equations, the translated equations are

$$q_i(\mathbf{p}, y - \Sigma\gamma_j p_j) + \gamma_i. \qquad (2)$$

For example, if $U(\mathbf{p}, y)$ is the simple homothetic utility function y/P, where P is a weighted geometric mean of prices, $\log P = \Sigma(\alpha_j \log p_j)$, the demand equations $q_i = \alpha_i y / p_i$ are translated to the famous Stone-Geary linear expenditure system (LES)

$$q_i = \gamma_i + \frac{\alpha_i}{p_i}(y - \Sigma\gamma_j p_j).$$

The idea of this paper is to replace (1) by the more general translation of $U(\mathbf{p}, y)$ to

$$U\left\{\mathbf{p}, y - y\sum\gamma_j\left(\frac{p_j}{y}\right)^{\beta_j}\right\}. \qquad (3)$$

This reduces to (1) if all $\beta_j = 1$. The condition $y > \Sigma\gamma_j p_j$ is replaced by

$$y > \sum\gamma_j p_j^{\beta_j} y^{1-\beta_j}\ . \qquad (4)$$

which will hold for positive β_j if y is not too small. If a β_i is negative, γ_i must also be negative.

This paper is particularly concerned with how generalised translation can produce demand systems with both good regularity and flexibility properties. Regularity means that, given appropriate ranges for the parameters, the indirect utility function complies with the constraints implied by rational economic behaviour.[1] Ideally, this should be possible for all prices and incomes (global regularity), but

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[1] That is, a consumer maximises direct utility under a budget constraint. This implies the indirect utility function $U(\mathbf{p}, y)$ should be homogeneous of degree zero in income y and prices **p**, non-decreasing in y, non-increasing in **p**, and convex or quasi-convex in **p.** These constraints imply

should at least hold for all values of these variables relevant to the situation under study. Flexibility is also required in that the corresponding demand system, while satisfying regularity, should be able to model a reasonably comprehensive spectrum of consumer behaviour - so the possible values of income and price elasticities, which are functions of parameter values, should not be seriously restricted[2]. The influential 'flexible functional forms' approach, employing Taylor series approximations to general utilitity (or cost) functions, sought systems embodying flexibility and hoped for regularity, but it seems that often their flexibility depends on their parameters being allowed to take values that contradict regularity[3]. Such models generally cannot test if observed consumption patterns do or do not accord with economic theory. The approach in this paper will be to start from globally regular systems and to improve their flexibility by generalised translation[4].

Properties of the general translated utility (3) are examined in section 2 and the corresponding demand system derived. Income and price elasticities are obtained and presented in terms of the elasticities of the parent system and the parameters of the generalised translation. Section 3 illustrates these results by considering a particular case, fairly parsimonious in parameters, that is interesting in its own right. Section 4 applies generalised translation to more parameter rich, though still globally regular, utility functions. In section 5 the possible application of generalised translation to non-globally regular utilities is discussed. Finally, in section 6, connections between generalised translation and Houthakker's (1960) indirect addilog system are explored[5].

II. TRANSLATED DEMAND EQUATIONS AND ELASTICITIES

Let

$$z = y - y\sum \gamma_j \left(\frac{p_j}{y} \right)^{\beta_j} ,$$

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corresponding conditions (aggregation, homogeneity, Slutsky symmetry and negativity) on the demand equations.

[2] More formal definitions of flexibility exist, differing in detail. See, for example, Diewert (1974).

[3] Caves and Christiansen (1980), Barnett and Lee (1985) and Cooper and McLaren (1992) have discussed the difficulty of reconciling flexibility and regularity for such systems.

[4] Of course, there are other approaches to improving regularity properties. Barnett (1983), Barnett and Lee (1985) and Chalfant (1986) reported wider regularity regions resulting from approximations based on Laurent, Muntz-Satz and Fourier expansions rather than Taylor series. Lewbel (1987) and Cooper and McLaren (1996) have also proposed systems.

[5] Lewbel (1985) has already used the term generalised translation in a paper on incorporating demographic effects into demand equations, which was a theme of considerable interest to Gorman. But the generalised translation of this paper is so obviously a generalisation of Gorman's that it seems inappropriate to call it anything else.

so that the translated utility is $U(\mathbf{p}, z)$. If $U(\mathbf{p}, y)$ is homogeneous of degree zero in income and prices, as it ought to be if a valid utility function, it is clear that $U(\mathbf{p}, z)$ is also.

$$\frac{\partial U(\mathbf{p},z)}{\partial y} = \frac{\partial U(\mathbf{p},z)}{\partial z}\frac{\partial z}{\partial y}$$

$$= \frac{\partial U(\mathbf{p},z)}{\partial z}\left\{1 - \sum \gamma_j \left(\frac{p_j}{y}\right)^{\beta_j} + \sum \gamma_j \beta_j \left(\frac{p_j}{y}\right)^{\beta_j}\right\} \qquad (5)$$

Assuming the original utility is non-decreasing in income, the derivative of $U(\mathbf{p}, z)$ with respect to z is non negative. Taking the terms within the chain brackets, equation (4) implies the first and second terms combined are positive while the third is also positive because, as previously mentioned, γ_i and β_i are presumed to have the same sign. So the translated utility is non-decreasing in income.

$$\frac{\partial U(\mathbf{p},z)}{\partial p_i} = \frac{\partial U}{\partial p_i} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial y}$$

$$= \frac{\partial U}{\partial p_i} + \frac{\partial U}{\partial z}\left\{-\gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta_i - 1}\right\} \qquad (6)$$

So assuming the original utility is non-increasing in prices, the translated utility is also. Proofs of the convexity of translated utilities are more complicated and depend on the forms of the original utilities. They are provided in Appendix 1 and show that above some income level the translated systems will be regular. This corresponds to the situation with Gorman translation. For example, the simple homothetic utility function y/P, $\log P = \Sigma(\alpha_j \log p_j)$, is globally convex in prices, while $(y - \Sigma\gamma_j p_j)/y$, which gives the LES, is convex if

$$\alpha_i (y - \Sigma \gamma_j p_j)^2 > 2\gamma_i^2 p_i^2 . \qquad (7)$$

So although Gorman translation increases flexibility by introducing extra parameters, it may invalidate regularity at very low incomes. This has not been seen as a difficulty, because, at least for analyses with time series data, interest focuses on inferences valid for recent or current time periods and income tends to increase with time. The term 'effective global regularity' has been applied to regularity everywhere except at low incomes[6]. Generalisations of (7) for generalised translation can be deduced

[6] The term seems due to Cooper and McLaren (1996).

from the results of Appendix 1 and will appear in later sections.

From Roy's lemma the demand equations are

$$q_{it}(\mathbf{p},y)=\frac{-\dfrac{\partial U(\mathbf{p},z)}{\partial p_i}/\dfrac{\partial U(\mathbf{p},z)}{\partial z}+\gamma_i\beta_i\left(\dfrac{p_i}{y}\right)^{\beta_i-1}}{1-\sum\gamma_j(1-\beta_{ji})\left(\dfrac{p_j}{y}\right)^{\beta_j}},$$

where q_{it} denotes the demand equation from the translated utility. Also by Roy's lemma,

$$-\frac{\partial U(\mathbf{p},z)}{\partial p_i}/\frac{\partial U(\mathbf{p},z)}{\partial z}=q_{io}(\mathbf{p},z)=q_{io}\left\{\mathbf{p},y-y\sum\gamma_j\left(\frac{p_j}{y}\right)^{\beta_j}\right\},$$

where q_{io} denotes the demand function derived from the original utility function. So

$$q_{it}(\mathbf{p},y)=\frac{q_{io}\left\{\mathbf{p},y-y\sum\gamma_j\left(\dfrac{p_j}{y}\right)^{\beta_j}\right\}+\gamma_i\beta_i\left(\dfrac{p_i}{y}\right)^{\beta_i-1}}{1-\sum\gamma_j(1-\beta_{ji})\left(\dfrac{p_j}{y}\right)^{\beta_j}},$$

or, more tidily,

$$q_{it}(\mathbf{p},y)=\frac{1}{V}\left\{q_{io}(\mathbf{p},z)+\gamma_i\beta_i\left(\frac{p_i}{y}\right)^{\beta_i-1}\right\},$$

or, in terms of budget shares

$$w_{it}(\mathbf{p},y)=\frac{1}{V}\left\{w_{io}(\mathbf{p},z)\frac{z}{y}+\gamma_i\beta_i\left(\frac{p_i}{y}\right)^{\beta_i}\right\}, \qquad (8)$$

where

$$V=1-\sum\gamma_j(1-\beta_j)\left(\frac{p_j}{y}\right)^{\beta_j}$$

and $w_{it}(\mathbf{p},y)$ is the budget share of the translated system as a function of **p** and y, and $w_{io}(\mathbf{p},z)$ is the budget share of the original system as a function of **p** and z. The income, own-price and cross-price elasticities of the translated system are derived in Appendix 2, in terms of $E_{io}(\mathbf{p,z}))$ and $e_{iko}(\mathbf{p},z)$, the income and price elasticities of the original system. They are:

$$E_{it}=\frac{w_{io}(\mathbf{p},z)}{w_{it}(\mathbf{p},y)}E_{io}(\mathbf{p},z)-\beta_i+\sum w_{jt}(\mathbf{p},y)\beta_j+\frac{z}{yV}\left\{1-\frac{w_{io}(\mathbf{p},z)}{w_{it}(\mathbf{p},y)}(1-\beta_i)-\sum w_{jo}(\mathbf{p},z)\beta_j\right\}$$

$$e_{iit} = e_{iio}(\mathbf{p},z) + \left[(1-\beta_i)\{w_{it}(\mathbf{p},y)-1\} - e_{iio}(\mathbf{p},z) - w_{io}(\mathbf{p},z)E_{io}(\mathbf{p},z)\right]\left(1 - \frac{w_{io}(\mathbf{p},z)}{w_{it}(\mathbf{p},y)}\frac{z}{yV}\right)$$

$$e_{ikt} = e_{iko}(\mathbf{p},z) + \left\{(1-\beta_k)w_{kt}(\mathbf{p},y) - e_{iko}(\mathbf{p},z) - w_{io}(\mathbf{p},z)E_{io}(\mathbf{p},z)\frac{w_{kt}(\mathbf{p},y)}{w_{it}(\mathbf{p},y)}\right\}\left(1 - \frac{w_{ko}(\mathbf{p},z)}{w_{kt}(\mathbf{p},y)}\frac{z}{yV}\right)$$

Even if the elasticities of the original system are very restricted, these elasticities are much less so. The Example of the next section illustrates this.

III GENERALISED TRANSLATION OF THE CONSTANT BUDGET SHARE MODEL

Returning to $U = y/P$, where $\log P = \Sigma(\alpha_j \log p_j)$, the original demand equations, in budget share form, are $w_{io} = \alpha_i$, or constant budget shares. Income and own price elasticity are unity and cross-price elasticities are zero. From (8) the translated demand equations are

$$w_i = \frac{\alpha_i\left\{1 - \sum \gamma_j \left(\frac{p_j}{y}\right)^{\beta_j}\right\} + \gamma_i\beta_i\left(\frac{p_i}{y}\right)^{\beta i}}{1 - \sum \gamma_j(1-\beta_j)\left(\frac{p_j}{y}\right)^{\beta_j}}$$

or, tidily,.

$$w_i = \frac{1}{V}\left\{\alpha_i \frac{z}{y} + \gamma_i\beta_i\left(\frac{p_i}{y}\right)^{\beta i}\right\} \tag{9}$$

It is shown in Appendix 1 that this system[7] will satisfy effective global regularity if

$$\alpha_i z^2 > \gamma_i\beta_i(\beta_i - 1)p_i\left(\frac{p_i}{y}\right)^{\beta_i - 1} z + 2\gamma_i^2\beta_i^2 p_i^2\left(\frac{p_i}{y}\right)^{2\beta_i - 2}, \tag{10}$$

which holds if $\alpha_i +, \Sigma\alpha_j = 1, \gamma_i\beta_i +$ and $\beta > -1$. It could even hold for an $\alpha_i = 0$ if $0 < \beta_i < 1$, although that may be purely academic. Obviously, (10) reduces to (7) if all $\beta_j = 1$.

If (9) is to have much practical value, it should be more flexible than the LES, which it becomes when all $\beta_j = 1$. The limitations of the LES include the linearity of all its Engel curves[8], its inability

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[7] This system has been examined in greater detail in Conniffe (2002a), but was not then understood as a case of generalised translation.

[8] One of the few generally agreed findings from empirical studies (see, for example, Lau, 1986) is that Engel curves, the relationships between expenditure and income at constant prices, are non-linear for at least some goods.

to cater for inferior goods and the tying of price effects to income aspects of price changes. In particular, the ratio of the cross-price elasticities e_{ik} and e_{jk} equals the ratio of income elasticities E_i and E_j, and the possibility of complementary goods is excluded.

From the formula of the previous section for translated income elasticity, dropping, for convenience, the subscripts indicating translated and original,

$$E_i = \frac{\alpha_i}{w_i} - \beta_i + \sum w_j \beta_j + \frac{z}{yV}\left\{1 - \sum \alpha_j \beta_j - \frac{\alpha_i (1-\beta_i)}{w_i}\right\}.$$

E_i is not generally a simple monotonic function of income. A commodity could be a luxury, or a necessity at different ranges of income, and could even be an inferior good (for a positive β_i and small α_i). However, as $y \to \infty$ the budget shares (9) tend to constancy and E_i to unity, which is not unreasonable and is a property of many other demand systems. That Engel curves can take a large variety of shapes is clear from the demand equation and the income elasticity, although as $y \to \infty$ they will approach linearity.

From the formulae of the previous section, the own-price and cross-price elasticities are

$$e_{ii} = -1 + \{\beta_i (1-w_i) + w_i - \alpha_i\}\left(1 - \frac{\alpha_i}{w_i}\frac{z}{yV}\right)$$

and

$$e_{ik} = \left(1 - \beta_k - \frac{\alpha_i}{w_i}\right)\left(w_k - \frac{z}{yV}\alpha_K\right),$$

so ratios of cross-price elasticities need not equal ratios of income elasticities. The compensated price effect

$$\frac{\partial q_i}{\partial p_k} + q_k \frac{\partial q_i}{\partial y}$$

is

$$\frac{q_i q_k}{y}\{1 - \beta_i - \beta_k + \Sigma w_j \beta_j + \frac{z}{yV}(1 - \Sigma \alpha_j \beta_j)\} + \frac{z}{yV}\{\frac{\alpha_i \alpha_k y}{p_i p_k} - \frac{q_i \alpha_k}{p_k}(1-\beta_k) - \frac{q_k \alpha_i}{p_i}(1-\beta_i)\}$$

and this can generally be negative or positive, allowing for complements as well as substitutes. So the extra n parameters contained in (9), relative to the LES, do substantially increase flexibility[9]. Of

[9] Pollak (1972) described a class of demand equations of the form $q_i = f_i(p_i / y, W)$, where W is a homogenous function of all prices and income, as exhibiting "generalised separability". Because all

course, any demand system with just 3n-1 parameters, will fall short of full flexibility if there are more than a few commodities. Sometimes, however, parsimony of parameters is desirable, especially if data are scarce, or observed price ranges do not embody sufficient independent variation to estimate a very detailed model of interacting price effects.

IV GENERALISED TRANSLATION OF LESS PARSIMONIOUS UTILITIES

Greater flexibility of price effects can be modelled by generalised translation of a less parsimonious, but still globally regular, original utility. Initially, a utility of the form y / P will be considered, where P is a function of more than n parameters, but is concave in prices.

It is easy to generate a globally regular homothetic utility function with 2n, or so[10], parameters from two utility functions y / P_1 and y / P_2, where P_1 and P_2, both functions of n parameters, are concave in prices. Then (Conniffe, 2002b) the sum, product and the reciprocal of reciprocals (harmonic mean) are also globally regular utility functions. For example, with $P_1 = \Pi p_j^{\alpha_j}$ and $P_2 = \Sigma \phi_j p_j$, with α_i and ϕ_i positive, the harmonic mean of the utility functions is

$$U = \frac{y}{\prod p_j^{\alpha_j} + \sum \phi_j p_j},$$

which is globally regular with 2n – 1 parameters. It gives the demand system

$$w_i = \frac{\alpha_i \Pi \mathrm{p}_j^{\alpha_j} + \gamma_i \mathrm{p}_i}{\Pi \mathrm{p}_j^{\alpha_j} + \Sigma \gamma_j \mathrm{p}_j},$$

which still, of course, gives a unitary income elasticity, but price elasticities of

$$e_{ii} = -w_i - \alpha_i (1 - \alpha_i) \frac{\Pi p_j^{\alpha_j}}{w_i (\Pi p_j^{\alpha_j} + \Sigma \phi_j p_j)}$$

and

$$e_{ik} = -w_k + \alpha_i \alpha_{ki} \frac{\Pi p_j^{\alpha_j}}{w_i (\Pi p_j^{\alpha_j} + \Sigma \phi_j p_j)}.$$

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prices, except p_i, take effect through W, there are implied symmetries in how commodity demands are affected by prices of other goods. The LES and the Indirect Addilog System (IAD) of Houthakker (1960) are of this class. But (9) is not, provided the α_j are non zero.

[10] Constraints such as $\Sigma \alpha_j = 0$ reduce parameters by one.

Generalised translation then produces a system with 4n-1 parameters, given by equation (8), with elasticities given by substituting the above into the formulae of section II. Many other systems are obtainable by other combinations of pairs of utility functions.

Combining simple utility functions is not the only way extra parameters could have been obtained. Original homothetic utility functions with more than 2n parameters are quite possible. For example,

$$U = \frac{y}{\sum \xi_{jk} p_j^{\frac{1}{2}} p_k^{\frac{1}{2}}},$$

could have been taken where the ξ_{jk} are positive and $\xi_{jk} = \xi_{kj}$ [11]. This has n(n+1)/2 parameters and gives the demand system

$$w_i = \frac{p_i^{\frac{1}{2}} \Sigma \xi_{ij} p_j^{\frac{1}{2}}}{2\Sigma\Sigma \xi_{jk} p_j^{\frac{1}{2}} p_k^{\frac{1}{2}}},$$

with price elasticities

$$e_{ii} = -\frac{1}{2} - w_i + \frac{p_i \xi_{ii}}{4w_i (\Sigma\Sigma \xi_{jk} p_j^{\frac{1}{2}} p_k^{\frac{1}{2}})^2}$$

and

$$e_{ik} = -w_k + \frac{p_i^{\frac{1}{2}} p_k^{\frac{1}{2}} \xi_{ik}}{4w_i (\Sigma\Sigma \xi_{jk} p_j^{\frac{1}{2}} p_k^{\frac{1}{2}})^2}.$$

In spite of a relative profusion of parameters this system still embodies restrictions on price effects as

$$\frac{\partial q_i}{\partial p_k} + q_k \frac{\partial q_i}{\partial y} = \frac{p_i^{-\frac{1}{2}} p_k^{-\frac{1}{2}} \xi_{ik}}{4(\Sigma\Sigma \xi_{jk} p_j^{\frac{1}{2}} p_k^{\frac{1}{2}})^2}$$

is positive if ξ_{ik} is, so ruling out complimentarity. So generalised translation, which gives a system with n(n+5)/2 parameters, will increases the flexibility of price effects as well as income effects. The demand equations and elasticities follow from the formulae of section II.

Appendix 1 shows that for functions of the form y / P, with P concave in prices, effective global regularity holds for the generalised translation if

[11] This is actually the homothetic case of the generalised Leontief utility function.

$$\varepsilon z^2 > \frac{\gamma_i \beta_i (\beta_i - 1)}{p_i} \left(\frac{p_i}{y} \right)^{\beta_i - 1} z + 2\gamma_i^2 \beta_i^2 \left(\frac{p_i}{y} \right)^{2\beta_i - 2} ., \tag{11}$$

where ε is the smallest eigenvalue of minus the Hessian of log P. This is the same as (10) with ε replacing α_i / p_i^2. Although ε is a function of prices and changes with the observations, it will be non-zero if P is strictly concave. Then, as before, $\gamma_i \beta_i +$ and $\beta > -1$ are all that is required. An ε of zero could be compatible with (11) if $0 < \beta_i < 1$.

There are, of course, globally regular utility functions that are not of the form y / P and that can involve even more parameters. The best known is probably the generalised Leontief

$$\left\{ 2 \sum \alpha_j \left(\frac{p_j}{y} \right)^{\frac{1}{2}} + \sum \sum \xi_{ij} \left(\frac{p_i}{y} \right)^{\frac{1}{2}} \left(\frac{p_j}{y} \right)^{\frac{1}{2}} \right\}^{-1}, \tag{12}$$

which is globally regular if all n(n+3)/2 parameters are positive. The demand equations are

$$w_i = \frac{\alpha_i \left(\frac{p_i}{y} \right)^{\frac{1}{2}} +_i \left(\frac{p_i}{y} \right)^{\frac{1}{2}} \sum \xi_{ij} \left(\frac{p_j}{y} \right)^{\frac{1}{2}}}{\sum \alpha_j \left(\frac{p_j}{y} \right)^{\frac{1}{2}} + \sum \sum \xi_{jk} \left(\frac{p_j}{y} \right)^{\frac{1}{2}} \left(\frac{p_k}{y} \right)^{\frac{1}{2}}}. \tag{13}$$

However, as various authors (Caves and Christiansen, 1980; Diewert and Wales, 1987) have remarked, these equations cannot represent some behaviourally plausible consumption patterns unless some parameters are permitted to become negative. This destroys regularity and so retaining Leontief parameters positive and translating to achieve flexibility is preferable. Appendix 1 shows that the generalised translation of (12) is effectively globally regular and the corresponding demand equations follow from (8) and (13).

V GENERALISED TRANSLATION OF NON GLOBALLY REGULAR UTILITIES

Although the main theme of this paper commences from globally regular utilities and uses translation to trade regularity at low income for flexibility and regularity at higher incomes, it is worth considering the impact of generalised translation on non globally regular original utilities. Some cases are clear enough, either in favour or against generalised translation.

A probably favourable case is generalised translation of y/P, with P a translog price index

$$\log P = \sum_j \alpha_j \log p_j + \sum_j \sum_k \xi_{jk} \log p_j \log p_k \text{, with } \xi_{jk} = \xi_{kj} .$$

This, the homothetic case of the general translog of Christensen, Jorgenson and Lau (1975), seems the natural extension of the geometric mean index, $\Sigma\alpha_j \log p_j$ and its generalised translation nests both the model of section III and the LES. However, y/P cannot be a globally regular utility function. Homogeneity requires

$$\sum_j \alpha_j = 1, \quad \sum_j \xi_{jk} = 0 \text{, for all k, and } \sum_k \xi_{jk} = 0 \text{ for all j,}$$

so some ξ_{jk} must be negative. But then

$$\frac{\partial \log P}{\partial p_i} = \frac{\alpha_i}{p_i} + \frac{1}{p_i}\sum \xi_{ij} \log p_j = \frac{S_i}{p_i} \text{, say,}$$

must become negative for some price combinations and

$$\frac{\partial U}{\partial p_i} = -\frac{y}{P^2}\frac{\partial P}{\partial p_i} = -\frac{y}{P}\frac{\partial \log P}{\partial p_i}$$

become positive[12]. However, perhaps the ‘real world’ prices occurring in the data, and the price ranges of particular interest for inference, are quite disjoint from these price combinations. Caves and Christiansen (1980) reported that regularity of the homothetic translog failed only at high relative price ratios. At least for fairly broad commodity definitions, perhaps price ratios may not reach such levels. Appendix 1 shows that if the S_i are positive and the matrix Ξ of estimated coefficients ξ_{jk} is negative semi-definite, regularity requires

$$S_i z^2 > \gamma_i \beta_i (\beta_i - 1) p_i \left(\frac{p_i}{y}\right)^{\beta_i - 1} z + 2\gamma_i^2 \beta_i^2 p_i^2 \left(\frac{p_i}{y}\right)^{2\beta_i - 2} ,$$

again the same as (10), with S_i replacing α_i, and the previous conditions applying to the β and γ parameters.. From (8) the translated demand equations are

$$w_i = \frac{1}{V}\left[\left\{\alpha_i + \sum \xi_{ij} \log p_j\right\}\frac{z}{y} + \gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta i}\right], \qquad (14)$$

and even without examining price elasticities, it is clear there is much more scope for cross-price

[12] Convexity of U with respect to prices would also fail.

effects than with the system (9), although at the cost of n(n-1)/2 extra parameters.

Some non globally regular systems cannot be improved by generalised translation. The AIDS model of Deaton and Muellbauer (1980a)

$$w_i = a_i + \sum_j c_{ij} \log p_j + b_i \log \frac{y}{P^*},$$

where P^* is a price index[13], has the utility function

$$U = \frac{\log y - \log P(\mathrm{p})}{B(\mathrm{p})},$$

with

$$\log B = b_0 + \sum b_j \log p_j, \quad \sum b_j = 0,$$

and

$$\log P = \sum_j \alpha_j \log p_j + \sum_j \sum_k c_{jk} \log p_j \log p_k, \quad c_{jk} = c_{kj}, \ \sum_j c_{jk} = \sum_k c_{jk} = 0.$$

This cannot be regular unless y is small, because B must decrease (and U increase) for prices with the negative b_j implied by $\Sigma b_j = 0$. Furthermore, convexity in prices demands that

$$\frac{\partial^2 U}{\partial p_i^2} = \frac{1}{p_i^2 B}\left\{ w_i(1+2b_i) - c_{ii} - b_i^2 \log \frac{y}{P} \right\}$$

be positive for all i, but it obviously becomes negative as y increases, particularly rapidly if b_i is substantially negative [14]. Clearly, if the original utility function loses its regularity with increasing income, the translated utility will also do so. Similar difficulties arise for some other demand systems, for example, Cooper and McLarens' (1992) MAIDS model[15] and for various 'rank 3' systems such as that of Ryan and Wales (1999).

VI GENERALISED TRANSLATION AND THE INDIRECT ADDILOG SYSTEM

There is another way of looking at generalised translation that can also commence from Gorman Translation. Gorman translation, can be thought of as producing weighted averages of interpretable budget shares. Applied to the constant budget share model it gave the LES

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[13] Strictly, $\log P^* = \log P = a_o + \sum_j a_j \log p_j + \frac{1}{2}\sum_j \sum_k c_{jk} \log p_j \log p_k$, but to retain linearity for estimation simplicity, this is often approximated by $\log P^* = \sum_j w_j \log p_j$.

[14] These are not original comments (see, for example, Cooper and McLaren , 1992).

[15] This modification of AIDS maintains regularity longer as income increases, but eventually loses it.

$$w_i = \alpha_i\left(1 - \frac{\sum \gamma_j p_j}{y}\right) + \frac{\gamma_i p_i}{y},$$

or

$$w_i = \frac{1}{y}\left\{\alpha_i\left(y - \sum \gamma_j p_j\right) + \frac{\gamma_i p_i}{\sum \gamma_j p_j}\sum \gamma_j p_j\right\},$$

which textbooks (e.g. Deaton & Muellbauer, 1980b, p.145) often interpret as giving a consumer's budget shares as a weighted average of a "rich person's" (constant budget share model) and a "poor person's" (Leontief, or constant ratios of quantities model) budget shares. Similarly, the generalised translation of the constant budget share model in section III gave

$$w_i = \frac{1}{V}\left\{\alpha_i \frac{z}{y} + \gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta i}\right\},$$

or

$$w_i = \frac{1}{V}\left\{\alpha_i \frac{z}{y} + \frac{\gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta i}}{\sum \gamma_j \beta_{ji} \left(\frac{p_j}{y}\right)^{ji}} \sum \gamma_j \beta_{ji} \left(\frac{p_j}{y}\right)^{ji}\right\}.$$

Now

$$\frac{\gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta i}}{\sum \gamma_j \beta_{ji} \left(\frac{p_j}{y}\right)^{ji}}$$

is the budget share of Houthakker's (1960) indirect addilog system. So w_i can be seen as a weighted average of a 'rich' person's and an 'IAD' person's budget shares, since

$$V = \frac{z}{y} + \sum \gamma_j \beta_{ji} \left(\frac{p_j}{y}\right)^{\beta_j}.$$

This way of viewing things helps explain of why generalised translation can give such a range of shapes of Engel curves, even when applied to a constant budget shares model , because Somermayer and Langhout (1972) demonstrated the great range of Engel curves arising from the IAD alone. More generally, (8)

$$w_{it}(\mathbf{p}, y) = \frac{1}{V}\left\{w_{io}(\mathbf{p}, z)\frac{z}{y} + \gamma_i\beta_i\left(\frac{p_i}{y}\right)^{\beta_i}\right\}$$

is a weighted average of the original system's budget share and the IAD budget share[16].

In Gorman translation $U(\mathbf{p}, y - \Sigma\gamma_j p_j)$ the γ_j are sometimes interpreted as subsistence quantities and $\Sigma\gamma_j p_j$ as subsistence income. While the translation parameters γ_j do not have to be understood in this way, the interpretation is sometimes attractive and useful. With generalised translation the analogue of subsistence income $\Sigma\gamma_j p_j$ is

$$y\sum\gamma_j\left(\frac{p_j}{y}\right)^{\beta_j},$$

which requires a wider interpretation than just subsistence income. For one thing it can change with income and assuming at least one β is less than unity, it increases to infinity as $y \to \infty$, with dominant term

$$\gamma_S p_S^{\beta_S} y^{1-\beta_S},$$

where β_s is the smallest of the β_i. But perhaps it can, at least sometimes, be interpreted as 'committed' income with the terms in the summation corresponding to a variety of types of commitment. For $\beta_i = 1$ say, the ith component of is $\gamma_i p_i$, so γ_i could be taken as a minimum essential quantity purchased at price p_i irrespective of income. For $\beta_k = 0$ say, the kth component of is $\gamma_k y$, so γ_k could be understood as a minimum proportion of income to be spent on commodity k irrespective of price. Intermediate interpretations are possible for a β between zero and one. A β greater than one, where the commodity fades out of committed income as income increases, could be interpreted as a less drastic option than the exclusion at all incomes that setting the corresponding

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[16] The IAD is a generalisation of the (simple) Leontief, to which it reduces if all $\beta_j = 1$. The IAD utility can itself be generalised to the generalised addilog:

$$-W = -\left\{\sum\gamma_j\left(\frac{p_j}{y}\right)^{\beta_j} + \sum\sum\theta_{jk}\left(\frac{p_j}{y}\right)^{\phi_j}\left(\frac{p_k}{y}\right)^{\phi_k}\right\}.$$

This suggests an even more general translation than (4), via $U\{\mathbf{p}, y - yW\}$. But unlike the Leontief or IAD, there are difficulties with the regularity of the generalised addilog, which would feed into the translated system, which could also be over parameterised.

$\gamma = 0$ would imply. It is true the case of a negative β and its corresponding γ seems incompatible with this committed income notion, but that situation also arises with subsistence income in Gorman translation when a γ is negative. These interpretations (and others) of committed income are all deducible from the Somermayer and Langhout (1972) interpretations of coefficients of the IAD demand system, which is unsurprising given that committed income is the product of y and the IAD indirect utility function.

Appendix 1: Convexity of the generalised translations

The derivative of $U(\mathbf{p}, z)$ with respect to p_i is

$$\frac{\partial U}{\partial z}\frac{\partial z}{\partial p_i} + \frac{\partial U}{\partial p_i}$$

and the second derivative with respect to p_i is

$$\frac{\partial^2 U}{\partial p_i^2} + \frac{\partial^2 U}{\partial z^2}\left(\frac{\partial z}{\partial p_i}\right)^2 + \frac{\partial U}{\partial z}\frac{\partial^2 z}{\partial p_i^2} + 2\frac{\partial^2 U}{\partial z \partial p_i}\frac{\partial z}{\partial p_i}. \tag{A1}$$

The second derivative with respect to p_i and p_j, remembering that

$$z = y - y\sum \gamma_j \left(\frac{p_j}{y}\right)^{\beta_j}$$

so that its second derivative with respect to p_i and p_j is zero, is

$$\frac{\partial^2 U}{\partial p_i \partial p_j} + \frac{\partial^2 U}{\partial z^2}\frac{\partial z}{\partial p_i}\frac{\partial z}{\partial p_j} + \frac{\partial^2 U}{\partial z \partial p_i}\frac{\partial z}{\partial p_j} + \frac{\partial^2 U}{\partial z \partial p_j}\frac{\partial z}{\partial p_i}. \tag{A2}$$

The Case $\mathbf{U = z / P}$

Matters simplify considerably when the original utility is income divided by a function of prices (which needs to be concave in prices if U is to be convex). Then

$$\frac{\partial U}{\partial z} = \frac{1}{P}, \quad \frac{\partial^2 U}{\partial z^2} = 0 \text{ and } \frac{\partial^2 U}{\partial z \partial p_i} = -\frac{1}{P^2}\frac{\partial P}{\partial p_i} = -\frac{1}{P}\frac{\partial \log P}{\partial p_i}.$$

Furthermore

$$\frac{\partial^2 U}{\partial p_i^2} = -\frac{z}{P}\frac{\partial^2 \log P}{\partial p_i^2} + \frac{\mathrm{z}}{\mathrm{P}}\left(\frac{\partial \log \mathrm{P}}{\partial \mathrm{p_i}}\right)^2 \; and \; \frac{\partial^2 U}{\partial p_i \partial p_j} = -\frac{z}{P}\frac{\partial^2 \log P}{\partial p_i \partial p_j} + \frac{\mathrm{z}}{\mathrm{P}}\left(\frac{\partial \log \mathrm{P}}{\partial \mathrm{p_i}}\right)\left(\frac{\partial \log \mathrm{P}}{\partial \mathrm{p_j}}\right)$$

and so (A1) and (A2) become

$$\frac{1}{P}\left\{-z\frac{\partial^2 \log P}{\partial p_i^2} + z\left(\frac{\partial \log \mathrm{P}}{\partial \mathrm{p_i}}\right)^2 - 2\left(\frac{\partial \log \mathrm{P}}{\partial \mathrm{p_i}}\right)\left(\frac{\partial \mathrm{z}}{\partial \mathrm{p_i}}\right) + \frac{\partial^2 z}{\partial p_i^2}\right\}$$

and

$$\frac{1}{P}\left\{-z\frac{\partial^2 \log P}{\partial p_i \partial p_j} + z\frac{\partial \log \mathrm{P}}{\partial \mathrm{p_i}}\frac{\partial \log \mathrm{P}}{\partial \mathrm{p_j}} - \frac{\partial \log \mathrm{P}}{\partial \mathrm{p_i}}\frac{\partial \mathrm{z}}{\partial \mathrm{p_j}} - \frac{\partial \log \mathrm{P}}{\partial \mathrm{p_j}}\frac{\partial \mathrm{z}}{\partial \mathrm{p_i}} + \right\}.$$

These can be rewritten as

$$\frac{1}{P}\left\{-z\frac{\partial^2 \log P}{\partial p_i^2} + \left(\sqrt{z}\frac{\partial \log P}{\partial p_i} - \frac{1}{\sqrt{z}}\frac{\partial z}{\partial p_i}\right)^2 + \frac{\partial^2 z}{\partial p_i^2} - \frac{1}{z}\left(\frac{\partial z}{\partial p_i}\right)^2\right\}$$

and

$$\frac{1}{P}\left\{-z\frac{\partial^2 \log P}{\partial p_i \partial p_j} + \left(\sqrt{z}\frac{\partial \log P}{\partial p_i} - \frac{1}{\sqrt{z}}\frac{\partial z}{\partial p_i}\right)\left(\sqrt{z}\frac{\partial \log P}{\partial p_j} - \frac{1}{\sqrt{z}}\frac{\partial z}{\partial p_j}\right) - \frac{1}{z}\left(\frac{\partial z}{\partial p_i}\right)\left(\frac{\partial z}{\partial p_j}\right)\right\}.$$

So the Hessian matrix (with respect to prices) of the translated utility is (apart from the $1/P$ multiplier) z by minus the Hessian matrix of $\log P$ with respect to prices, which is positive definite[17], plus GG', which, is positive semi-definite, where G is the vector with ith term

$$\sqrt{z}\frac{\partial \log P}{\partial p_i} - \frac{1}{\sqrt{z}}\frac{\partial z}{\partial p_i},$$

plus a diagonal matrix with ith term

$$\frac{\partial^2 z}{\partial p_i^2} = -\frac{\gamma_i \beta_i (\beta_i - 1)}{p_i}\left(\frac{p_i}{y}\right)^{\beta_i - 1}$$

minus HH'/z, where H is the vector with ith term

$$\frac{\partial z}{\partial p_i} = -\gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta_i - 1}.$$

While $-HH'$ is negative semi-definite, the addition to it of the diagonal matrix with ith term

[17] If P is concave, $\log P$ is concave.

$$2\left(\frac{\partial z}{\partial p_i}\right)^2 = 2\gamma_i^2\beta_i^2\left(\frac{p_i}{y}\right)^{2\beta_i-2}$$

gives a nonnegative definite matrix (diagonals positive and principal minors of higher order zero). So the translated utility is convex with respect to prices if the negative of the Hessian of log P multiplied by z minus the diagonal matrix with elements

$$\frac{\gamma_i\beta_i(\beta_i-1)\left(\frac{p_i}{y}\right)^{\beta_i-1}}{p_i}+\frac{2\gamma_i^2\beta_i^2\left(\frac{p_i}{y}\right)^{2\beta_i-2}}{z} \qquad \text{(A3)}$$

is nonnegative definite.

Convexity for the constant budget share model

Now let $\log P = \Sigma(\alpha_j \log p_j)$. The Hessian of log P with respect to prices is diagonal. So U is convex with respect to prices if

$$\frac{\alpha_i z}{p_i^2}-\frac{\gamma_i\beta_i(\beta_i-1)\left(\frac{p_i}{y}\right)^{\beta_i-1}}{p_i}-\frac{2\gamma_i^2\beta_i^2\left(\frac{p_i}{y}\right)^{2\beta_i-2}}{z}$$

is positive, or

$$\alpha_i z^2 > \gamma_i\beta_i(\beta_i-1)p_i\left(\frac{p_i}{y}\right)^{\beta_i-1} z + 2\gamma_i^2\beta_i^2 p_i^2\left(\frac{p_i}{y}\right)^{2\beta_i-2}, \qquad \text{(A4)}$$

which is equation (10) of section 3. If all the β_j are positive, the left hand side is of order y^2, while the right hand side terms are of order $y^{2-\beta_i}$ and $y^{2-2\beta_i}$, so since α_i is positive the condition holds provided y (and hence z) is not small. The first right hand side term is negative if $0 < \beta_i < 1$, so that a zero α_i might even be compatible with convexity in that situation. For at least one of the β_j negative, let β_s be the smallest (most negative). The greatest power of y in z is then $y^{1-\beta_s}$ with coefficient $|\gamma_s| p_s^{\beta_s}$ (γ_s is negative if β_s is). So the left hand side is of order $y^{2-2\beta_s}$, while the right hand terms are of order $y^{2-\beta_i-\beta_s}$ and $y^{2-2\beta_i}$ and it is clear the critical term is i= s. Comparing coefficients of $y^{2-2\beta_s}$, the condition is $\alpha_s - \beta_s$ greater than β_s^2. This will be true if $0 > \beta_s > -1$, even if α_s is small (or even zero). So effective global regularity holds[18] if $\alpha_i +, \Sigma\alpha_j = 1, \gamma_i \beta_i +$ and $\beta > -1$.

Convexity with P concave in prices

The translated utility is again convex with respect to prices if the negative of the Hessian of log P multiplied by z minus (A3) is nonnegative definite. Let the smallest eigenvalue of minus the Hessian matrix of log P, that is, the matrix with i,k th term

$$-\frac{\partial^2 \log P}{\partial p_i \partial p_k}$$

be ε. If log P is strictly concave minus the Hessian is positive definite and can be written equal to Q+ε I, where I is the identity matrix, Q is positive semi-definite and ε is positive. Then the condition for regularity is

$$\varepsilon z^2 > \frac{\gamma_i \beta_i (\beta_i - 1)}{p_i}\left(\frac{p_i}{y}\right)^{\beta_i - 1} z + 2\gamma_i^2 \beta_i^2 \left(\frac{p_i}{y}\right)^{2\beta_i - 2} . \qquad \text{(A5)}$$

This seems the same as (A4) with ε replacing α_i / p_i^2, but it is actually a much stricter condition. Its equivalent, where α_s denotes the smallest α_i would have been

$$\alpha_s z^2 > \gamma_i \beta_i (\beta_i - 1) p_i \left(\frac{p_i}{y}\right)^{\beta_i - 1} z + 2\gamma_i^2 \beta_i^2 p_i^2 \left(\frac{p_i}{y}\right)^{2\beta_i - 2} ,$$

for all i and not just i=s. So (A5) probably exaggerates the income required for effective global regularity, although as the left hand side is of order y^2 (assuming $\beta_i +$), while the right hand side terms are of order $y^{2-\beta_i}$ and $y^{2-2\beta_i}$, there is no doubt it will hold as y increases. Less demanding conditions could be derived for specific cases of P. Again, if ε =0 for some observations (log P not strictly concave) effective global regularity could still hold if $0 < \beta_i < 1$, although the case is probably not of practical interest. For negative β the same argument as before gives the condition

$$\varepsilon p_s^2 - \beta_s > \beta_s^2$$

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[18] The conditions on the γ_j and β_j correspond to those for the regularity of the utility function of the indirect addilog system. The validity conditions of the IAD have been debated in the literature more than once, as the exchanges between Gamelatos (1973, 1974) and Somermayer (1974), and between Akin and Stewart (1979) and Murty (1982) testify.

and again this will be true if 0 > β_s > -1.

Convexity with P a translog price index.

As before, convexity requires the negative of the Hessian of log P multiplied by z minus A(3) to be nonnegative definite. The negative of the Hessian of Log P is

$$-L\Xi L + \Lambda ,$$

where L is the diagonal matrix with ith element $1/p_i$, Ξ is the matrix of coefficients ξ_{jk} and Λ is the diagonal matrix with ith element S_i / p_i^2. If, following estimation of the ξ_{jk}, Ξ is found to be concave the convexity requirement becomes

$$S_i z^2 > \gamma_i \beta_i (\beta_i - 1) p_i \left(\frac{p_i}{y} \right)^{\beta_i - 1} z + 2\gamma_i^2 \beta_i^2 p_i^2 \left(\frac{p_i}{y} \right)^{2\beta_i - 2} .$$

This gives the condition of section V. The condition is the same as (A4) with S_i replacing α_i and the same arguments apply as regards the required ranges of β_i and γ_i. But while the α_i were constants, the S_i are function of prices and the validity of the argument depends on their being positive.

Convexity with a Generalised Leontief utility

It is now necessary to return to (A1) and (A2) to take account of terms that no longer vanish. However, much of the previous approach will still apply. The (translated) utility function is

$$U = \left\{ 2\sum \alpha_j \left(\frac{p_j}{z} \right)^{\frac{1}{2}} + \sum\sum \xi_{ij} \left(\frac{p_i}{z} \right)^{\frac{1}{2}} \left(\frac{p_j}{z} \right)^{\frac{1}{2}} \right\}^{-1}$$

and could be written in the form z / P^*, where

$$P^* = 2z^{\frac{1}{2}} \sum \alpha_{ij} p_i^{\frac{1}{2}} + \sum\sum \xi_{ij} p_i^{\frac{1}{2}} p_j^{\frac{1}{2}} ,$$

although P^* is now a function of income, as well as prices. The convexity of the original utility with respect to prices must imply that that P^* is concave in prices and therefore $\log P^*$ is also.

$$\frac{\partial^2 U}{\partial p_i^2} = -\frac{z}{P^*}\frac{\partial^2 \log P*}{\partial p_i^2} + \frac{\mathrm{z}}{P^*}\left(\frac{\partial \log P^*}{\partial \mathrm{p_i}}\right)^2$$

and

$$\frac{\partial^2 U}{\partial z \partial p_i} = -\frac{1}{P^*}\frac{\partial \log P*}{\partial p_i} + \frac{\mathrm{z}}{\left(P^*\right)^2}\left(\frac{\partial \log P^*}{\partial \mathrm{z}}\right)\left(\frac{\partial \log P^*}{\partial \mathrm{p_i}}\right) - \frac{z}{P^*}\frac{\partial^2 \log P*}{\partial z \partial p_i}$$

$$= -\frac{1}{P^*}\frac{\partial \log P*}{\partial p_i} + C_i \text{, say.} \qquad \text{(A6)}$$

Although the first term of (A6), which is easily shown to be

$$-\frac{\alpha_i}{2(P^*)^2}\left(\frac{z}{p_i}\right)^{\frac{1}{2}}$$

is of order $1/\sqrt{z}$, the term C_i, which may be shown to be

$$-\frac{\alpha_i \sqrt{z} + \sum \xi_{ij} p_j^{\frac{1}{2}}}{(P^*)^3 p_i^{\frac{1}{2}}} \sum\sum \xi_{ij} p_i^{\frac{1}{2}} p_j^{\frac{1}{2}}$$

is of order $1/z$. This will be important later. Returning to (A1), two of its terms

$$\frac{\partial^2 U}{\partial p_i^2} + 2\frac{\partial^2 U}{\partial z \partial p_i}\frac{\partial z}{\partial p_i}$$

now give

$$-\frac{z}{P^*}\frac{\partial^2 \log P*}{\partial p_i^2} + \frac{1}{P^*}\left(\sqrt{z}\frac{\partial \log P^*}{\partial \mathrm{p_i}} - \frac{1}{\sqrt{z}}\frac{\partial z}{\partial p_i}\right)^2 - \frac{1}{P*z}\left(\frac{\partial z}{\partial p_i}\right)^2 + 2C_i\frac{\partial z}{\partial p_i}.$$

The rest of (A1) is

$$\frac{\partial^2 U}{\partial z^2}\left(\frac{\partial z}{\partial p_i}\right)^2 + \frac{\partial U}{\partial z}\frac{\partial^2 z}{\partial p_i^2}. \qquad \text{(A7)}$$

Similarly, terms from (A2)

$$\frac{\partial^2 U}{\partial p_i \partial p_j} + \frac{\partial^2 U}{\partial z \partial p_i}\frac{\partial z}{\partial p_j} + \frac{\partial^2 U}{\partial z \partial p_j}\frac{\partial z}{\partial p_i}$$

give

$$-\frac{z}{P^*}\frac{\partial^2 \log P*}{\partial p_i \partial p_j} + \frac{1}{P^*}\left(\sqrt{z}\frac{\partial \log P^*}{\partial \mathrm{p_i}} - \frac{1}{\sqrt{z}}\frac{\partial z}{\partial p_i}\right)\left(\sqrt{z}\frac{\partial \log P^*}{\partial \mathrm{p_j}} - \frac{1}{\sqrt{z}}\frac{\partial z}{\partial p_j}\right) - \frac{1}{P*z}\frac{\partial z}{\partial p_i}\frac{\partial z}{\partial p_j}$$

plus

$$C_i\frac{\partial z}{\partial p_j} + C_j\frac{\partial z}{\partial p_i}$$

and, from the remaining term of (A2),

$$\frac{\partial^2 U}{\partial z^2}\frac{\partial z}{\partial p_i}\frac{\partial z}{\partial p_j}. \tag{A8}$$

Some terms are almost identical to those occurring in the z/P case, with P^* instead of P, and, as before, can be expressed as following from the sum of a positive definite matrix (minus z/P^* by the Hessian of $\log P^*$) plus two positive semi-definite matrices minus two diagonal matrices with diagonal terms

$$\frac{2}{P^* z}\left(\frac{\partial z}{\partial p_i}\right)^2 \quad \text{and} \quad -\frac{\partial U}{\partial z}\frac{\partial^2 z}{\partial p_i^2}.$$

It is worth noting that these terms are respectively of order $1/2-2\beta_i$ and $1/2-\beta_i$ in income.

Turning to the extra terms involved, those in C_i and C_j can be seen as following from

$$LL' - P^* zCC' - \frac{1}{P^* z}\left(\frac{\partial z}{\partial \mathbf{p}}\right)\left(\frac{\partial z}{\partial \mathbf{p}}\right)', \tag{A9}$$

where L is the vector with elements

$$\sqrt{P^* z}C_i + \frac{1}{\sqrt{P^* z}}\frac{\partial z}{\partial p_i},$$

and C is the vector with elements C_i. The first matrix in (A9) is positive semi-definite. The second is negative semi-definite, but becomes positive semi-definite if the diagonal matrix with diagonal terms $2P^* zC_i^2$ is added to it. The same applies to the third term if the diagonal matrix with diagonal terms

$$\frac{2}{P^* z}\left(\frac{\partial z}{\partial p_i}\right)^2$$

is added. These diagonal matrices have to be subtracted again, of course.

The first term in (A7) and the (A8) term arise from the matrix

$$\frac{\partial^2 U}{\partial z^2}\left(\frac{\partial z}{\partial \mathbf{p}}\right)\left(\frac{\partial z}{\partial \mathbf{p}}\right)'$$

and since

$$\frac{\partial^2 U}{\partial z^2} = -\frac{\sum \alpha_j p_j^{\frac{1}{2}}}{\left(P^*\right)^3}\left(\sum \alpha_j p_j^{\frac{1}{2}} + \frac{3}{2\sqrt{z}}\sum\sum \xi_{ij} p_i^{\frac{1}{2}} p_j^{\frac{1}{2}}\right) \qquad \text{(A10)}$$

is negative the matrix is negative semi-definite. Adding the diagonal matrix with diagonal terms

$$-2\frac{\partial^2 U}{\partial z^2}\left(\frac{\partial z}{\partial p_i}\right)^2$$

produces a positive semi-definite matrix. Note that (A10) is of order $z^{-\frac{3}{2}}$.

So the ith element of the final subtracted diagonal matrix is

$$\left(\frac{4}{P^* z} - \frac{\partial^2 U}{\partial z^2}\right)\left(\frac{\partial z}{\partial p_i}\right)^2 - \frac{\partial U}{\partial z}\frac{\partial^2 z}{\partial p_i^2} + 2P^* z C_i^2, \qquad \text{(A11)}$$

where, of course,

$$\frac{\partial z}{\partial p_i} = -\gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta_i - 1} \quad \text{and} \quad \frac{\partial^2 z}{\partial p_i^2} = -\frac{\gamma_i \beta_i (\beta_i - 1)}{p_i}\left(\frac{p_i}{y}\right)^{\beta_i - 1}.$$

From earlier comments it is clear the terms of (A11), assuming the β_i positive, are respectively of order

$$y^{\frac{1}{2} - 2\beta_i}, \quad y^{\frac{1}{2} - \beta_i} \quad \text{and} \quad y^{-\frac{1}{2}}.$$

As already said, minus the Hessian matrix of $\log P^*$, multiplied by z / P^*, is positive definite in prices and it is easily seen to be of order $\sqrt{z}$ or $\sqrt{y}$. So the same arguments as previously will demonstrate effective global regularity.

Appendix 2: Derivation of income and own price elasticities of translated systems

For convenience of notation, the dependence of quantities on prices and income will not be indicated explicitly except for z, the translated income.

The income elasticity is

$$E_{it} = \frac{y}{q_{it}}\frac{\partial q_{it}}{\partial y} = \frac{y}{q_{it}}\frac{\partial}{\partial y}\left[\frac{1}{V}\left\{q_{io}(z) + \gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta_i - 1}\right\}\right].$$

$$\frac{\partial q_{it}}{\partial y} = \frac{1}{V}\left\{\frac{\partial q_{io}(z)}{\partial z}\frac{\partial z}{\partial y} + \frac{\gamma_i \beta_i (1 - \beta_i)}{y}\left(\frac{p_i}{y}\right)^{\beta_i - 1}\right\} - \frac{q_{it}}{V}\frac{\partial V}{\partial y}.$$

Noting that

$$\frac{\partial z}{\partial y} = V \quad \text{and} \quad \frac{\partial V}{\partial y} = \frac{1}{y}\sum \gamma_j \beta_j (1-\beta_j)\left(\frac{p_j}{y}\right)^{\beta_j}$$

this is

$$\frac{\partial q_{io}(z)}{\partial z} + \frac{\gamma_i \beta_i (1-\beta_i)}{yV}\left(\frac{p_i}{y}\right)^{\beta_i - 1} - q_{it}\frac{1}{yV}\sum \gamma_j \beta_j (1-\beta_i)\left(\frac{p_i}{y}\right)^{\beta_i}.$$

So

$$E_{it} = \frac{y}{q_{it}}\frac{\partial q_{io}(z)}{\partial z} + \frac{\gamma_i \beta_i (1-\beta_i)}{q_{it} V}\left(\frac{p_i}{y}\right)^{\beta_i - 1} - \frac{1}{V}\sum \gamma_j \beta_j (1-\beta_i)\left(\frac{p_i}{y}\right)^{\beta_i}.$$

Using

$$\gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta_i} = \frac{p_i}{y}\gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta_i - 1} = \frac{p_i}{y}\{q_{it}V - q_{io}(z\} = w_{it}V - \frac{z}{y}w_{io}(z),$$

where the w_{it} and $w_{io}(z)$ are the budget shares, and that

$$E_{io}(z) = \frac{z}{q_{io}(z)}\frac{\partial q_{io}(z)}{\partial z}$$

E_{it} becomes

$$\frac{w_{io}(z)}{w_{it}}E_{io}(z) - \beta_i + \sum w_{jt}\beta_j + \frac{z}{yV}\left\{1 - \frac{w_{io}(z)}{w_{it}(y)}(1-\beta_i) - \sum w_{jo}(z)\beta_j\right\}.$$

The own-price elasticity is

$$e_{iit} = \frac{p_i}{q_{it}}\frac{\partial q_{it}}{\partial p_i} = \frac{p_i}{q_{it}}\frac{\partial}{\partial p_i}\left[\frac{1}{V}\left\{q_{io}(z) + \gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta_i - 1}\right\}\right].$$

$$\frac{\partial q_{it}}{\partial p_i} = \frac{1}{V}\left\{\frac{\partial q_{io}(z)}{\partial p_i} + \frac{\partial q_{io}(z)}{\partial z}\frac{\partial z}{\partial p_i} - \frac{\gamma_i \beta_i (1-\beta_i)}{y}\left(\frac{p_i}{y}\right)^{\beta_i - 2}\right\} - \frac{q_{it}}{V}\frac{\partial V}{\partial p_i}.$$

Noting that

$$\frac{\partial z}{\partial p_i} = -\gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta_i - 1} \quad \text{and} \quad \frac{\partial V}{\partial p_i} = -\frac{1}{y}\gamma_i \beta_i (1-\beta_i)\left(\frac{p_i}{y}\right)^{\beta i - 1},$$

this is

$$\left\{\frac{1}{V}\frac{\partial q_{io}(z)}{\partial p_i} - \frac{\partial q_{io}(z)}{\partial z}\{q_{it} - \frac{q_{io}(z)}{V}\} - \frac{(1-\beta_i)}{p_i}\{q_{it} - \frac{q_{io}(z)}{V}\}\right\} + \frac{q_{it}(1-\beta_i)\{q_{it} - \frac{q_{io}(z)}{V}\}}{y}.$$

So e_{iit} equals

$$\frac{q_{io}(z)}{q_{it}V}e_{iio}(z)-\left\{\frac{p_i q_{io}(z)}{q_{it}z}E_{io}(z)+\frac{(1-\beta_i)}{q_{it}}-\frac{p_i(1-\beta_i)}{y}\right\}\left\{q_{it}-\frac{q_{io}(z)}{V}\right\},$$

or

$$e_{iit}=e_{iio}(z)+\{(1-\beta_i)(w_{it}-1)-e_{iio}(z)-w_{io}(z)E_{io}(z)\}\left(1-\frac{w_{io}(z)}{w_{it}}\frac{z}{yV}\right).$$

The cross-price elasticity is

$$e_{ikt}=\frac{p_k}{q_{it}}\frac{\partial q_{it}}{\partial p_k}=\frac{p_k}{q_{it}V}\left\{\frac{\partial q_{io}(z)}{\partial p_k}+\frac{\partial q_{io}(z)}{\partial z}\frac{\partial z}{\partial p_k}\right\}-\frac{p_k}{V}\frac{\partial V}{\partial p_k}$$

$$=\frac{q_{io}(z)}{q_{it}V}e_{iko}(z)-\left\{\frac{p_k q_{kt} q_{io}(z)}{q_{it}z}E_{io}(z)-(1-\beta_k)\frac{p_k q_{kt}}{y}\right\}\left(1-\frac{q_{ko}(z)}{q_{kt}V}\right)$$

or

$$e_{ikt}=e_{iko}(z)+\left\{(1-\beta_k)w_{kt}-e_{iko}(z)-w_{io}(z)E_{io}(z)\frac{w_{kt}}{w_{it}}\right\}\left(1-\frac{w_{ko}(z)}{w_{kt}}\frac{z}{yV}\right).$$

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