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When does 'All Eggs in One Risky Basket' Make Sense?

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Abstract: In an important paper comparing expected utility and mean-variance analysis, Feldstein (1969) examined a simple portfolio problem involving just two assets, one riskless and one risky. He concluded there could easily be 'plunging', that is, investment in the risky asset alone. His background assumptions were that the risky asset's yield was log normally distributed and that the investor's attitude to risk was expressible by a logarithmic utility. We look at how conclusions are affected by choice of distribution and utility function. While conclusions can depend on choice of distribution, they are remarkably robust to choice within the range of plausible positive distributions. In contrast, conclusions are sensitive to choice of utility function and we find the key determinant to be how much the investor's relative risk aversion differs from unity and in what direction. Based on historical stock market returns, our analysis implies that the prevalence of diversification that is observed is consistent with a relative risk aversion coefficient of about 2.5.

I INTRODUCTION

Feldstein (1969) published an important and much cited paper criticising the use of mean-variance (or $\mu - \sigma$) analysis rather than expected utility in the theory of economic behaviour under uncertainty. He based his analysis on a simple portfolio problem involving just two assets, one riskless and one risky. But since the risky asset could itself be considered to be the optimal, or market, portfolio of risky assets, the separation theorem of mean-variance portfolio analysis suggests the importance of the problem.

There were three components to Feldstein's paper. The first outlined the possibility that the $\mu - \sigma$ indifference curves of a risk averter might not be convex downwards. The second argued that an investor choosing a combination of a riskless and a risky asset might be much more likely to opt or 'plunge' for a risk-only portfolio than previous analysis had suggested, so that, contrary to Tobin's (1958) analysis, diversification between money and bonds need not generally follow from risk aversion. The third criticised the use of $\mu - \sigma$ analysis for multi-asset portfolios. Subsequent discussion of Feldstein's paper has concentrated on the first and third components. The second, which appears to have received relatively little attention, provides the motivation for this paper.

Feldstein assumed a log normal distribution[1] of the return on the risky asset and a logarithmic utility function. He demonstrated that a utility-maximising strategy would lead to plunging for the risky asset given values of the parameters of the log normal that he considered far from unlikely. We will review his results in section II along with some discussion of subsequent comments. We continue to examine how much plunging depends on the specification of the probability distribution and utility function. We show in section III that if the logarithmic utility function is retained, Feldstein's results are remarkably robust across a range of distributions. However, in section IV, we show that the situation is very different with regard to choice of utility function and we relate the variations in the likelihood of plunging to the degree of risk aversion embedded in the utility functions. Finally, in section V we consider some implications of these findings.

II FELDSTEIN'S ANALYSIS

Feldstein, following Tobin (1958), considered an investor with initial wealth A, allocating a proportion p to the risky asset, so that, after one period, wealth becomes y = (1-p)A + pAx, where the riskless asset is assumed non-interest bearing and x is random with a mean presumed greater than unity[2]. Feldstein took x as log normal with parameters μ^* and σ^* , that is, a density of

$$f(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma^*}\frac{1}{x}e^{-\frac{(\log x-\mu^*)^2}{2\sigma^{*2}}}, \quad 0 < x < \infty. \tag{1}$$

Then the mean of x

$$\mu = e^{\mu^* + \frac{1}{2}\sigma^{*2}}$$

and the variance of x

$$\sigma^2 = e^{2\mu^* + \sigma^{*2}}\left(e^{\sigma^2} - 1\right).$$

For investment in the risky asset to have any attraction, Feldstein further assumed that the investor is risk averse with a logarithmic utility function. So the investor wants to choose p so as to

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[1] Since, as is well known, expected utility and $\mu - \sigma$ analysis are compatible given normality, Feldstein needed to assume some non-normal distribution. However, log normality, or other positive distributions, seem more plausible than the normal and have more empirical support.

[2] Obviously, the mathematical nature of the problem is unchanged by attaching a non-stochastic yield to the riskless asset and adjusting the yield of the risky asset accordingly.

maximise

$$E\{u(y)\} = E[\log\{(1-p)A + pAx\}]$$

$$= \log A + E[\log\{1 + p(x-1)\}]\,. \qquad (2)$$

The derivative

$$\frac{\partial E(u)}{\partial p} = E\left\{\frac{x-1}{1+p(x-1)}\right\}$$

is obviously positive at p = 0, so that there will certainly be some investment in the risky asset. At p = 1 it is

$$1 - E\left(\frac{1}{x}\right) = 1 - \int \frac{1}{x} f(x)dx \qquad (3)$$

which is

$$1 - e^{-\mu^* + \frac{1}{2}\sigma^{*2}} = 1 - \frac{1}{\mu}\left(1 + \frac{\sigma^2}{\mu^2}\right).$$

So the derivative is still positive, implying plunging, that is, all funds invested in the risky asset, if

$$\mu > \left(1 + \frac{\sigma^2}{\mu^2}\right). \qquad (4)$$

Feldstein maintained this would often be the case, giving the example of $\mu = 1.05$, when plunging occurs unless σ is well more than four times the expected bond yield of 5%. He contrasted this situation with the ubiquity of diversification implied by Tobin's (1958) mean variance analysis of the same two asset model and linked it to non convexity of indifference curves, so as to emphasise that mean variance analysis is not always an adequate substitute for expected utility maximisation.

While many subsequent papers, whether defending or further criticising mean-variance analysis, have discussed Feldstein's paper (including, among others, Tobin (1969), Tsiang (1972), Bierwag (1974), Borch (1974), Levy (1974), Mayshar (1978) Felstein (1978) and Meyer (1987)), the plunging phenomenon itself has not featured prominently. Mayshar accepted the correctness of Feldstein's analysis of plunging, but maintained it was really unrelated to issues of the convexity of indifference curves, or validity of mean-variance analysis. One of his arguments was that Feldstein had analysed indifference curves generated by $E\{u(x)\}$ when x is assumed log normal, but in the portfolio problem considered a variable $y = A(1 + p(x-1)\}$, which being a translation of a log normal has a somewhat different distribution. Another was the more general point that if the investor is restricted to choosing values of p and will obtain the same value of x irrespective of choice of p, then whatever the distribution of x, the distributions of the variables

$$\frac{y(p)-\mu_y(p)}{\sigma_y(p)}$$

are obviously identical, where $\mu_y(p) = A\{1+p(\mu-1)\}$ and $\sigma_y(p) = Ap\sigma$. Then whatever the distribution and whatever the (concave) utility function, the optimal p, assuming it < 1, could be obtained from mean variance analysis in terms of μ_y and σ_y. While more roundabout than direct determination by equating the derivative of $E\{u(y)\}$ with respect to p to zero, it is certainly compatible with it[3]. But whether p is < 1, or whether plunging occurs is ignored in this argument and it is not at all clear that it should be. There would seem no sense in using mean variance analysis when p=1 and, if plunging is possible, interest should focus on how distribution and utility function affect its occurrence. Certainly, condition (4) was derived assuming log normality and log utility. Note too that if p = 1, y is log normally distributed if x is. In contrast to Mayshar, Meyer (1987) disputed the validity of Feldstein's result on plunging, claiming it resulted from an implicit borrowing restriction in the model. But if the investor could borrow without cost, this would just increase A with plunging still occurring.

So no detailed examination seems to have been conducted into how the occurrence of plunging might change with the distribution or how it might depend on the precise utility function chosen.

III VARYING THE PROBABILITY DISTRIBUTION

Retaining, for the present, the logarithmic utility, formulae (2) and (3) remain unchanged. Plunging will occur if

$$1-E\left(\frac{1}{x}\right) > 0, \tag{5}$$

while a mixture of assets will be optimal if it is negative. If x could take negative values, the expectation of the reciprocal of x may not exist, which can usually be interpreted as $E \to \infty$, implying that plunging cannot occur. So, for example, assuming a normal distribution for x would

[3] Other authors have made this point including (implicitly) Bierwag (1974) and Meyer (1987).

exclude plunging. But if we assume that the most that can be lost by investment in a risky asset is the amount invested, a non-negative distribution is appropriate for x.

Taking x as positively distributed , the well known delta method for approximation of expectations proceeds

$$E\left(\frac{1}{x}\right)=E\left(\frac{1}{\mu+\varepsilon}\right)=\frac{1}{\mu}E\left\{\left(1+\frac{\varepsilon}{\mu}\right)^{-1}\right\}=\frac{1}{\mu}E\left\{1-\frac{\varepsilon}{\mu}+\frac{\varepsilon^2}{\mu^2}-\frac{\varepsilon^3}{\mu^3}+\ .\ .\ .\right\}$$

$$=\frac{1}{\mu}+\frac{\sigma^2}{\mu^3}-\frac{\mu_3}{\mu^4}+\ .\ .\ .,\qquad (6)$$

where μ_3 denotes the third moment about the mean. Now if we ignore terms with denominators involving powers of μ greater than 3, this gives

$$E\left(\frac{1}{x}\right)\approx\frac{1}{\mu}\left(1+\frac{\sigma^2}{\mu^2}\right),$$

which implies (4) again as the determinant of plunging. Admittedly, this approximation assumes that the coefficient of variation is not large, but note that if the distribution is positively skew, implying μ_3 positive, this reduces (6) making it more likely that (5) is positive and plunging occurs.

However, we need not rely on approximation arguments to show that plunging can occur for other than the log normal. Suppose x follows a Pearson Type 3 distribution (often called the two parameter gamma)

$$f(x)=\frac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-\frac{x}{\beta}}\ ,\qquad 0<x<\infty$$

where the mean and variance can be shown to be $\mu=\alpha\beta$ and $\sigma^2=\alpha\beta^2$. Then

$$E\left(\frac{1}{x}\right)=\frac{1}{\beta(\alpha-1)}=\frac{1}{\mu-\dfrac{\sigma^2}{\mu}}.$$

This will be less than unity and plunging will occur if

$$\mu>1+\sigma^2/\mu.\qquad (7)$$

So plunging is quite plausible unless variation is very large, although not quite as likely as with a log normal since

$$\frac{1}{\mu-\frac{\sigma^2}{\mu}}=\frac{1}{\mu}\left(1-\frac{\sigma^2}{\mu^2}\right)^{-1}=\frac{1}{\mu}\left(1+\frac{\sigma^2}{\mu^2}+\frac{\sigma^4}{\mu^4}+\ .\ .\ .\right)>\frac{1}{\mu}\left(1+\frac{\sigma^2}{\mu^2}\right).$$

However, taking $\mu=1.05$ as in Feldstein's example, (7) shows plunging occurs unless $\sigma>.22$, again more than least four times the expected bond yield of 5%.

Again, suppose x follows a Weibull distribution, which, like the log normal and Gamma distributions, has been found in various empirical studies to provide a good fit to wealth distributions. The density is

$$f(x)=\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}},\qquad 0<x<\infty.$$

It is evident that f(x)/x tends to infinity as x goes to zero for $\alpha<2$, implying that the expectation goes to infinity also and that plunging cannot occur. For $\alpha\geq 2$ exact integration gives

$$E\left(\frac{1}{x}\right)=\frac{1}{\beta}\Gamma\left(1-\frac{1}{\alpha}\right). \tag{8}$$

The mean of a Weibull is

$$\mu=\beta\,\Gamma\left(1+\frac{1}{\alpha}\right) \tag{9}$$

and so for large α, since $\Gamma(1)=1$, the right hand side of (8) becomes the reciprocal of μ. This must be less than unity, as μ was presumed greater than unity, so plunging must then occur. Practically plausible values of α are best judged from the coefficient of variation. The variance of a Weibull is

$$\sigma^2=\beta^2\left\{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma^2\left(1+\frac{1}{\alpha}\right)\right\}$$

and, obviously, large α implies small σ. The square of the coefficient of variation is

$$\frac{\sigma^2}{\mu^2}=\frac{\Gamma\left(1+\frac{2}{\alpha}\right)}{\Gamma^2\left(1+\frac{1}{\alpha}\right)}-1. \tag{10}$$

So for any μ and σ, α can be deduced from (10) and then β from (9), so that (8) can show if plunging occurs. Taking the case of $\mu=1.05$ and σ = .2, or four times the expected yield, we find (8) takes the value .996 so that plunging occurs even with that much variation, which is essentially the same as Feldstein found for the log normal distribution.

Similar findings follow for other plausible positive distributions for which algebraic exact solutions are obtainable. The multiple of the expected yield that variation must exceed before plunging is ruled out can vary somewhat, but the overall situation is clear. Plunging ought not to be an infrequent prediction with distributions that are usually considered plausible candidates for wealth, at least when we believe an investor's risk aversion can be represented by logarithmic utility.

IV VARYING THE UTILITY FUNCTION

To point up the difference that choice of utility function can make, we first contrast logarithmic utility with the extremely popular negative exponential utility

$$u(y) = 1 - e^{-\lambda y}\text{, with } \lambda > 0\text{,}$$

where, as before, y = (1-p)A + pAx. This is often taken in conjunction with a normal distribution for x and then it is well known that then the optimal p is

$$p = \frac{\mu - 1}{\lambda A \sigma^2}$$

which must be less than unity if A is large enough and so the amount invested in the risky asset is Ap, constant irrespective of the amount available for investment. However, this absence of plunging also occurs with the distributions considered in the previous sections. Taking the two-parameter gamma, for example, the expected utility is

$$E\{u(y)\} = 1 - \frac{1}{\beta^\alpha \Gamma(\alpha)} \int e^{-\lambda A\{1+p(x-1)\}} x^{\alpha-1} e^{-\frac{x}{\beta}} dx\,.$$

$$= 1 - \frac{e^{-\lambda A(1-p)}}{\beta^\alpha \Gamma(\alpha)} \int e^{-x(\lambda Ap + \frac{1}{\beta})} e^{-\frac{x}{\beta}} x^{\alpha-1} dx\,.$$

This can be integrated exactly to give

$$1 - \frac{e^{-\lambda A(1-p)}}{(\lambda Ap\beta + 1)^\alpha}\,.$$

Differentiating with respect to p gives

$$\frac{\lambda A e^{-\lambda A(1-p)}}{(\lambda Ap\beta + 1)^\alpha} \left\{ \frac{\alpha\beta}{\lambda Ap\beta + 1} - 1 \right\}$$

which, if A is large, will obviously be negative at p = 1. The optimum investment Ap is

$$Ap = \frac{\alpha\beta - 1}{\lambda\beta} = \frac{\mu(\mu - 1)}{\lambda\sigma^2},$$

again a constant, irrespective of A. The contrast between results for the logarithmic utility and the negative exponential utility must be due to the different degrees of risk aversion they embody. The coefficients of absolute risk aversion and relative risk aversion are

$$R = \frac{1}{y} \quad \text{and} \quad R_r = 1, \qquad \text{for} \quad u = \log y$$

and

$$R = \lambda \quad \text{and} \quad R_r = \lambda y, \qquad \text{for} \quad u = 1 - e^{-\lambda y}.$$

Furthermore $u = \log y$ is unbounded and $\to \infty$ as y $\to \infty$, while the negative exponential utility $\to 1$ as y $\to \infty$. There is so much less to be gained from large increments in wealth with the negative exponential utility.

For a finer exploration of the influence of risk aversion on plunging, we take the utility function

$$u = 1 - \lambda y^{-\lambda}, \quad \text{with } \lambda \geq -1.$$

For λ negative this is unbounded $\to \infty$ as y $\to \infty$ and for λ positive it is bounded $\to 1$ as y $\to \infty$. In fact, $\lambda = 0$ corresponds to $u = \log y$ and, obviously, $\lambda = -1$ gives risk neutrality. The coefficients of risk aversion are

$$R = \frac{\lambda + 1}{y} \qquad \text{and} \qquad R_r = \lambda + 1,$$

showing decreasing absolute risk aversion and constant relative risk aversion. So this utility function is not at all as risk averse as the negative exponential utility, but it can be either less risk averse than the logarithmic utility (if λ is negative) or more risk averse (if λ is positive). So it should be a sensitive criterion for examining plunging. For any distribution density f(x), the expected utility is

$$E\{u(y)\} = 1 - \int \lambda \left[A\{1 + p(x-1)\}\right]^{-\lambda} f(x)dx.$$

Differentiating with respect to p and setting p = 1 gives

$$\lambda^2 A^{-\lambda} \int x^{-\lambda-1}(x-1)f(x)dx. \qquad (11)$$

If this is positive plunging occurs. It can be evaluated by integration for any of the positive distributions mentioned earlier. For (1), the log normal density, (11) becomes

$$\lambda^2 A^{-\lambda} e^{-\lambda\mu^* + \frac{\lambda^2}{2}\sigma^{*2}} \left\{ 1 - e^{-\mu^* + (\lambda + \frac{1}{2})\sigma^{*2}} \right\}.$$

So plunging occurs if

$$\mu^* > (\lambda + \frac{1}{2})\sigma^{*2},$$

or expressed in of μ and σ

$$\mu > \left(1 + \frac{\sigma^2}{\mu^2}\right)^{\lambda+1}, \qquad (12)$$

which reduces to Feldstein's criterion (4) if $\lambda = 0$. For λ negative, plunging is even more likely than with logarithmic utility, in the sense that it will occur in spite of ever greater variability. At $\lambda = -1$ plunging is certain. Conversely, for λ positive and increasing, plunging becomes less likely and will not occur if λ is large enough. So, since decreasing absolute risk aversion and constant relative risk aversion holds for this utility function whatever the value of λ, it is not these characteristics in themselves that determine plunging. Rather, since $R_r = \lambda + 1$, it is the difference from unity (and direction) of the relative risk aversion, which of course is the (negative) elasticity of the slope of the utility function. High R_r implies absence of plunging.

These findings are not dependent on the log normal alone. Returning to (11), but assuming f(x) follows the two parameter gamma, gives

$$\frac{\lambda^2 A^{-\lambda}}{\beta^\alpha \Gamma(\alpha)} \int x^{-\lambda-1}(x-1)x^{\alpha-1} e^{-\frac{x}{\beta}} dx \; .$$

Integration gives

$$\frac{\lambda^2 A^{-\lambda}}{\beta^{\lambda+1}\Gamma(\alpha)} \{\beta\Gamma(\alpha-\lambda) - \Gamma(\alpha-\lambda-1)\}.$$

$$= \frac{\lambda^2 A^{-\lambda}\Gamma(\alpha-\lambda-1)}{\beta^{\lambda+1}\Gamma(\alpha)} \{\beta(\alpha-\lambda-1) - 1\}$$

So plunging occurs if $\beta(\alpha - \lambda - 1) > 1$, or in terms of μ and σ

$$\mu > 1 + \frac{\sigma^2}{\mu}(\lambda + 1). \qquad (13)$$

This becomes (7) for $\lambda = 0$ and $R_r = \lambda + 1$ plays the same role as it does for the log normal distribution. Similar results follow for the Weibull, although formulae are somewhat messier.

An approximation type argument is also be worth developing. For any utility function

$$\frac{\partial E(u)}{\partial p} = \int u'[A\{1 + p(x-1)\}]f(x)dx$$

and at p = 1 this is

$$\left(\frac{\partial E}{\partial p}\right)_{p=1} = A\int u'(Ax)(x-1)f(x)dx$$

By Taylor expansion

$$u'(Ax) = u'(A\mu) + A(x-\mu)u''(A\mu) + \frac{1}{2}A^2(x-\mu)^2u'''(A\mu) + \ . \ . \ .$$

or

$$u'(Ax) \approx u'(A\mu)\{1 - A(x-\mu)u'R(A\mu)\} + \frac{1}{2}A^2(x-\mu)^2u'''(A\mu).$$

By substitution

$$\left(\frac{\partial E}{\partial p}\right)_{p=1} \approx Au'(A\mu)\left\{(\mu-1) - \frac{\sigma^2}{\mu}R_r(A\mu)\right\} + \frac{A^3}{2}u'''(A\mu)\left\{\mu_3 + (\mu-1)\sigma^2\right\},$$

where μ_3 is the third central moment of the distribution. The first term on the right hand side is positive if

$$\mu > 1 + \frac{\sigma^2}{\mu}R_r(A\mu),$$

and clearly (13) is the case of this corresponding to $u = 1 - \lambda e^{-\lambda y}$ and the two parameter gamma distribution. The second term will be positive when u''' is, as it often is, and the distribution is positively skew, which would favour plunging. However, the terms involving higher derivatives in the Taylor expansion, neglected in the above expansion, might partially, or even totally, counterbalance this term, which must be occurring with $u = 1 - \lambda e^{-\lambda y}$ and the two parameter gamma.

So the overall finding is that the closeness to unity of relative risk aversion is the key parameter as regards plunging. Values near unity, or less than it, imply plunging unless variation is very large relative to yield. Large positive values of R_r imply absence of plunging. Returning to the negative exponential, or constant absolute risk aversion utility, with which we commenced this section, its $R_r \to \infty$ as y $\to \infty$, fully in line with the absence of plunging found for that utility function.

V DISCUSSION

The previous sections would lead one to expect that plunging ought to be a reasonably frequently observed phenomenon. But that does not seem to be the real world situation. It is well known that fund managers, even when the portfolio is ostensibly 'aggressively managed', always hold appreciable quantities of riskless assets. So is there something wrong with the theoretical assumptions or other aspects of the model and analysis?

We would get diversification and no plunging with different distributional assumptions, for example, if we replace positive distributions like the lognormal by the normal. But there is plenty of literature reporting empirical tests of normality that rejected that distribution, but found positively skewed distributions such as the Weibull or log normal to be a better fit to data sets. So the probabilistic or distributional assumptions seem unlikely to provide an explanation for the discrepancy between theory and practice. Of course, even Feldstein (1969) agreed that plunging will not occur if σ is sufficiently greater than μ. Could it be that the relative magnitudes considered in previous sections are quite different from those experienced by fund managers? However, they do not seem out of line with the long-run evidence. If we look at a broad market index like the Standard and Poor's 500 for the period 1881 to 1999, the standard deviation of returns is about 3.2 times the annual return, well short of the ratio required for diversification according to Feldstein.

Is the model structure over simplified? The problem of investing in just two assets, one risky and one riskless, is, of course, just about the simplest possible. But it is a standard textbook starting point. Textbook treatment typically ignores any possibility of plunging, assuming that optimisation is attained with the derivative of expected utility zero. Furthermore, the comparative statics of the choice problem are often examined by approaches that depend for their development on the very existence of the zero derivative . The treatment in Varian (1992, 184-186) provides an example. The model is also historically important. Tobin (1958) used it to examine the behaviour of $\mu - \sigma$ investors and in particular to illustrate the effect of taxing capital gains from a risky asset. Feldstein (1969) employed it also, of course, as did commentators on his work. Recently, Ormiston and Schlee (2001) have undertaken further analyses of the comparative static properties of this model to reveal how investors respond to some special changes in distribution. Although mentioning that some conditions are required for an "interior solution", which is equivalent to absence of plunging, they do not pursue this matter, but conduct analyses on the assumption they are satisfied. So many authors have considered the model relevant in spite of its simplicity. Also, intuitively, the two-asset problem ought to be suggestive for the multiple asset case. The idea, mentioned earlier, of identifying the risky asset with the market portfolio would turn the simple problem into the choice of optimal portfolio, but perhaps we should stop short of this[4].

This would seem to leave only the risk aversity of fund managers as the explanatory factor. As was shown in section IV, high relative risk aversion would imply diversification. It seems strange though to attribute high relative risk aversion to 'aggressive' fund managers. But perhaps there is another, usually neglected, source of uncertainty associated with the formulation of the model. Feldstein and ourselves assume μ and σ, and indeed the distribution, known. The uncertainty analysed arose only from the randomness generated within that distribution. But a fund manager would only have estimates of parameters such as μ and σ. It is true that in many econometric problems the uncertainty introduced through estimation errors of parameters is negligible compared to that inherent in the stochastic nature of the model, because the estimates are relatively precise being based on adequate data. Although data on asset prices are usually abundant, arguments are frequently made that parameter structures may be non-stationary, so that most of the data may be of doubtful value for estimating current values of parameters. If fund managers have, at best, tentative estimates about which they are none too confident, this uncertainty may reinforce risk aversion.

Our analysis indicates that the likelihood of plunging diminishes as the relative risk aversion coefficient deviates positively from unity for given values of μ and σ. An implication of this result is that, for given values of μ and σ, we can establish a lower bound to the value of relative risk aversion which would be consistent with the ubiquity of diversification. Thus if we again take the S&P 500 data for the period 1881 to 1999, the implied lower bound for R_r is about 2.5. If we exclude the bull-run period from 1982 on, the implied lower bound is about 2.

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[4] It could be argued that the separation theorem is a product of $\mu - \sigma$ analysis and lacks validity in an expected utility context.

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