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## Generalised Means of Simple Utility Functions with Risk Aversion

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*Abstract:* The paper examines the properties of a generalised mean of simple utilities each displaying risk aversion, that is, with first derivative positive and second derivative negative. It proves the mean is itself a valid utility function with the appropriate signs for derivatives and investigates risk aversion properties. It shows that simple component utilities, each of which may have quite restricted risk aversion properties, can be parsimoniously combined through the generalised mean formula to give a much more versatile utility function.

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# Generalised Means of Simple Utility Functions with Risk Aversion

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## I INTRODUCTION

The generalised mean of  $n$  values  $x_1, x_2, \dots, x_n$  is

$$\left[ \sum_{j=1}^n w_j x_j^{-\lambda} \right]^{-\frac{1}{\lambda}},$$

with  $\sum w_j = 1$ . The formula is familiar in economics as giving a constant elasticity of substitution function. Taking  $w_j = 1/n$  and  $\lambda = -1$  gives the standard arithmetic mean  $\bar{x}$ , while taking  $\lambda = 0$  gives (via the usual limiting argument as  $\lambda \rightarrow 0$ ) the geometric mean

$$\prod_{j=1}^n x_j^{\frac{1}{n}}$$

and  $\lambda = 1$  the harmonic mean

$$\frac{n}{\sum_{j=1}^n \frac{1}{x_j}}.$$

This paper considers the risk aversion properties of a generalised mean of utility functions of wealth  $y$

$$U(y) = \left[ \sum_{j=1}^n w_j u_j^{-\lambda}(y) \right]^{-\frac{1}{\lambda}}, \quad (1)$$

where each function  $u_j(y)$  is increasing and concave in  $y$ . The idea here is that simple component utilities  $u_j(y)$ , each of which may have quite restricted risk aversion properties, can be combined

through the generalised mean formula to give a much more versatile utility function<sup>1</sup>. Formula (1) resembles those occurring in procedures to construct regular indirect utility functions appropriate for commodity consumer demand studies (Conniffe, 2002, 2007). However, then matters are much more complicated as utilities are functions of income and all commodity prices. Here  $U(Y)$  is univariate and interest centres on the properties of the Arrow-Pratt coefficients of absolute and relative risk aversion

$$R_A(y) = -\frac{U''(y)}{U'(y)} \quad \text{and} \quad R_R(y) = -\frac{yU''(y)}{U'(y)}.$$

## II CONCAVITY AND RISK AVERSION OF $U(y)$

First we need to show that (1) is a valid utility function if all of the  $u_j(y)$  are. That is, given each first derivative  $u'_j(y)$  is positive and each second derivative  $u''_j(y) \leq 0$ , then  $U'(y)$  is positive and  $U''(y) \leq 0$ . Differentiating (1)

$$U'(y) = \left[ \sum w_j u_j^{-\lambda} \right]^{\frac{1}{\lambda}-1} \left[ \sum w_j u_j^{-\lambda-1} u'_j \right], \quad (2)$$

which is positive. Differentiating again

$$U''(y) = \left[ \sum w_j u_j^{-\lambda} \right]^{\frac{1}{\lambda}-1} \left[ \sum w_j u_j^{-\lambda-1} u''_j \right] - (\lambda + 1) \left[ \sum w_j u_j^{-\lambda} \right]^{\frac{1}{\lambda}-2} \left[ \sum w_j u_j^{-\lambda} \sum w_j u_j^{-\lambda-2} u_j'^2 - \left( \sum w_j u_j^{-\lambda-1} u'_j \right)^2 \right] \quad (3)$$

and the top term on the right hand side of equation (3) is clearly negative. The sign of the bottom term is less obvious. Let

$$x_j^2 = w_j u_j^{-\lambda} \quad \text{and} \quad z_j = u'_j / u_j.$$

Then the bottom term becomes

$$-(\lambda + 1) \left[ \sum x_j^2 \right]^{\frac{1}{\lambda}-2} \left\{ \sum x_j^2 \sum (x_j z_j)^2 - \left[ \sum x_j (x_j z_j) \right]^2 \right\}$$

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<sup>1</sup> For non-integer values of  $\lambda$  or  $1/\lambda$  imaginary values of (1) could take imaginary values if a  $u_j(y)$  is negative. Then  $u_j(y)$  should be redefined positive. For example, the exponential utility can be written  $1 - \exp(-\gamma y)$  rather than  $-\exp(-\gamma y)$ . Use of  $\log y$  could assume the unit of measurement chosen so that  $y > 1$ . The power function can be written  $y^{1-\alpha}$  for  $0 \leq \alpha < 1$ , but  $1 - y^{1-\alpha}$  for  $\alpha > 1$ , again with  $y > 1$ .

and by the Cauchy-Schwarz inequality the term in chain brackets is positive. So provided  $\lambda \geq -1$ ,  $U(y)$  is a valid utility function.

From (2) and (3) the Arrow-Pratt coefficient of absolute risk aversion,  $R_A$ , is

$$\begin{aligned} & -\frac{\sum w_j u_j^{-\lambda} \frac{u''_j}{u_j}}{\sum w_j u_j^{-\lambda} \frac{u'_j}{u_j}} + (\lambda + 1) \left[ \frac{\sum w_j u_j^{-\lambda} \left( \frac{u'_j}{u_j} \right)^2}{\sum w_j u_j^{-\lambda} \frac{u'_j}{u_j}} - \frac{\sum w_j u_j^{-\lambda} \frac{u'_j}{u_j}}{\sum w_j u_j^{-\lambda}} \right] \\ & = \frac{\sum w_j u_j^{-\lambda} R_{Aj} \frac{u'_j}{u_j}}{\sum w_j u_j^{-\lambda} \frac{u'_j}{u_j}} + (\lambda + 1) \left[ \frac{\sum w_j u_j^{-\lambda} \left( \frac{u'_j}{u_j} \right)^2}{\sum w_j u_j^{-\lambda} \frac{u'_j}{u_j}} - \frac{\sum w_j u_j^{-\lambda} \frac{u'_j}{u_j}}{\sum w_j u_j^{-\lambda}} \right], \end{aligned} \quad (4)$$

where  $R_{Aj}$  is the coefficient of absolute risk aversion for the  $j$ th utility function. Since the coefficient of relative risk aversion  $R_R$  is just defined as  $yR_A$ , its formula is easily derived from (4) and is

$$R_R = \frac{\sum w_j u_j^{-\lambda} R_{Rj} \frac{u'_j}{u_j}}{\sum w_j u_j^{-\lambda} \frac{u'_j}{u_j}} + (\lambda + 1)y \left[ \frac{\sum w_j u_j^{-\lambda} \left( \frac{u'_j}{u_j} \right)^2}{\sum w_j u_j^{-\lambda} \frac{u'_j}{u_j}} - \frac{\sum w_j u_j^{-\lambda} \frac{u'_j}{u_j}}{\sum w_j u_j^{-\lambda}} \right] \quad (5)$$

Some special cases and deductions from (4) and (5) are interesting. For  $\lambda = -1$  the formula for absolute risk aversion becomes

$$\frac{\sum w_j R_{Aj} u'_j}{\sum w_j u'_j}.$$

Differentiating gives

$$\begin{aligned} R'_A & = \frac{\sum w_j R'_{Aj} u'_j}{\sum w_j u'_j} + \frac{\sum w_j R_{Aj} u''_j}{\sum w_j u'_j} - \frac{(\sum w_j R_{Aj} u'_j) \sum w_j u''_j}{(\sum w_j u'_j)^2} \\ & = \frac{\sum w_j R'_{Aj} u'_j}{\sum w_j u'_j} - \left[ \frac{\sum w_j R_{Aj}^2 u'_j}{\sum w_j u'_j} - \frac{(\sum w_j R_{Aj} u'_j)^2}{(\sum w_j u'_j)^2} \right], \end{aligned} \quad (6)$$

Putting  $w_j R_{Aj}^2 u'_j = x_j^2$  and  $w_j u'_j = z_j^2$ , the Cauchy-Schwarz inequality shows that the term in square brackets is positive. So if each utility function has constant absolute risk aversion (CARA) or decreasing absolute risk aversion (DARA),  $U(y)$  certainly shows DARA. In particular, for a weighted sum of exponential utilities,  $u_j(y) = 1 - \exp(-\gamma_j y)$ , each with constant  $R_{Aj} = \gamma_j$

$$R_A = \frac{\sum w_j \gamma_j^2 e^{-\gamma_j y}}{\sum w_j \gamma_j e^{-\gamma_j y}},$$

which decreases as wealth increases unless the  $\gamma_j$  are all equal. The fundamental paper about risk aversion by Pratt (1964), while largely about a single utility function, did consider the sum of two utilities and deduced the persistence of DARA just described.

From (5) the coefficient of relative risk aversion is

$$R_R = \frac{\sum w_j R_{Rj} u'_j}{\sum w_j u'_j}$$

and its derivative is

$$R'_R = \frac{\sum w_j R'_{Rj} u'_j}{\sum w_j u'_j} - \frac{1}{y} \left[ \frac{\sum w_j R_{Rj}^2 u'_j}{\sum w_j u'_j} - \frac{(\sum w_j R_{Rj} u'_j)^2}{(\sum w_j u'_j)^2} \right]$$

and the argument made about (6) applies again so that if each utility function has constant relative risk aversion (CRRA) or decreasing relative risk aversion (DRRA),  $U(y)$  certainly displays DRRA. In particular, a weighted sum of power utilities, each with constant  $R_{Rj} = \alpha_j$  displays DRRA unless the  $\alpha_j$  are all equal.

But these results depend on  $\lambda = -1$ , that is on (1) being a weighted sum of utilities. For other values of  $\lambda$  matters can be more complicated although there are some simple results. For example, with  $\lambda$  zero, (5) becomes

$$R_R = \frac{\sum w_j R_{Rj} \frac{u'_j}{u_j}}{\sum w_j \frac{u'_j}{u_j}} + (\lambda + 1)y \left[ \frac{\sum w_j \left( \frac{u'_j}{u_j} \right)^2}{\sum w_j \frac{u'_j}{u_j}} - \sum w_j \frac{u'_j}{u_j} \right].$$

When each utility is of the power form, CRRA holds for each with  $R_{Rj} = \alpha_j$  and also each  $u'_j / u_j$  is

proportional to  $1/y$ . So  $y$  cancels out of the formula for  $R_R$  and CRRA holds for  $U(y)$  also, unlike the situation for  $\lambda = -1$ . For  $\lambda = +1$  it can be shown that  $R_R(y)$  can increase with  $y$  even if  $R_{Rj} = \alpha_j$  unless the  $\alpha_j$  are all equal. Although these investigations of how risk aversion properties common to all component utilities are or are not preserved in  $U(y)$  could be taken a lot further, it is probably not worth while doing so. The reason was mentioned in the introduction. If the object of combination is to derive a more versatile utility function than its components, combination of utilities with very different risk aversion properties, rather than with similar ones, seems desirable.

### III COMBINING COMPONENTS WITH DISSIMILAR RISK PROPERTIES

We will usually want reasonable parsimony of parameters, so probably only two, or at most three, utility functions would be combined. So we commence with a very parsimonious example. The utility function  $u_1(y) = \Phi(y)$ , where  $\Phi(y)$  denotes the (cumulative) distribution of the standard normal, has no parameters<sup>2</sup>. Since  $\Phi'(y) = \phi(y)$ , where  $\phi(y)$  is the standard normal density, and  $\Phi''(y) = -y\phi(y)$  the simple  $R_{A1} = y$  results, of course implying increasing absolute risk aversion (IARA). For another parameter-free utility  $u_2(y) = \log y$  it is well known that  $R_{A2} = 1/y$ , which implies DARA<sup>3</sup>. It is interesting to examine what results from combining utilities with absolute aversions respectively increasing and decreasing in proportion to  $y$ . Choosing  $\lambda = -1$  gives

$$U(y) = w\Phi(y) + (1-w)\log y, \quad (7)$$

a utility function with one parameter  $w$ . Either directly or from the formulae of the previous section

$$R_A = \frac{wy^3\phi(y) + 1 - w}{wy^2\phi(y) + (1-w)y}.$$

For fixed  $y$

$$\frac{\partial R_A}{\partial w} = \frac{(y^2 - 1)\phi(y)}{(wy\phi(y) + (1-w))^2}$$

which is positive for  $y > 1$ , so that relative risk aversion increases with  $w$  as would be expected. For

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<sup>2</sup>  $\Phi(y)$  is concave only for  $y > 0$ , but that is no difficulty when  $y$  is income or wealth.

<sup>3</sup> Assuming  $y > 1$  to ensure positive utility.

fixed  $w$  and putting  $\theta = w/(1-w)$

$$R'_A = \frac{\partial R_A}{\partial y} = \frac{\theta^2 y^4 \phi^2(y) - \theta y \phi(y)(y^2 - 1)(y^2 - 2) - 1}{y^2 (\theta y \phi(y) + 1)^2} \quad (8)$$

Since the third term in the numerator of (8) is  $-1$ , it is obvious that DARA holds if  $w$ , and therefore  $\theta$ , is small enough. But since the second term is negative if  $y > \sqrt{2}$ , DARA can hold even if  $\theta$  is not particularly small provided  $y$  is large enough. If  $\theta$  is large ( $w$  near 1) the derivative is clearly positive and IARA holds with  $R'_A \rightarrow 1$  as  $\theta \rightarrow \infty$ . However, unless  $\theta$  is huge  $R'_A$  can become negative for large  $y$  because  $\phi(y)$  is then very small<sup>4</sup>. So this combination of  $u_1$  and  $u_2$  via (1) can display either IARA or DARA depending on choice of  $w$  and show transitions from one state to another with increasing wealth  $y$ . The point of transition could be obtained by setting the numerator of (8) to zero, but because of the  $\phi(y)$  terms the equation is complicated.

Combination of the same utilities with  $\lambda = 0$  gives

$$U(y) = \Phi^w(y) (\log y)^{1-w} \quad (9)$$

and rather tedious differentiation can show the risk aversion properties are similar to those just outlined for (7). However, note that  $\Phi(y)$  approaches unity for large  $y$  in both (7) and (9). So, for example, (9) is then effectively

$$U(y) = (\log y)^{1-w}$$

and it is easily verified that this utility has

$$R_A = \frac{w}{\log y} + \frac{1}{y},$$

which is DARA as expected. Actually  $y$  does not have to be particularly large for this to hold since  $\Phi(3.72) = .9999$  and so the transition from IARA to DARA occurs at quite low  $y$ . This could be remedied, if desired, by introducing a standard deviation parameter  $\sigma$  to the cumulative normal distribution  $u_1(y)$ . Then  $u_1(3.72\sigma) = .9999$  and data sets implying a higher transition point can be adequately fitted by estimating a  $\sigma$  considerably greater than unity.

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<sup>4</sup> Since the  $r$ th moment of the normal exists  $Lt_{y \rightarrow \infty} y^r \phi(y) \rightarrow 0$ .

Taking  $\lambda = 1$  would give the harmonic mean utility

$$U(y) = \frac{1}{\frac{w}{\Phi(y)} + \frac{1-w}{\log y}}$$

and there are many other possibilities. A two-parameter utility with  $\sigma = 1$  would be given by

$$U(y) = \left[ w\Phi(y)^{-\lambda} + (1-w)(\log y)^{-\lambda} \right]^{\frac{1}{\lambda}}$$

with the obvious possible extension to a three-parameter utility by allowing a  $\sigma$  parameter. Again,  $w$  could be set to a predetermined value, retaining a two parameter function in  $\lambda$  and  $\sigma$ .

Probably a more familiar utility displaying increasing absolute risk aversion is the quadratic  $u_3(y) = y - by^2$ , where  $b$  is presumed small, so that the requirement  $y < 1/2b$  does not limit the range of  $y$  too greatly. This could be combined with  $u_2(y) = \log y$  to again obtain utilities capable of displaying both IARA and DARA. A three parameter function is

$$U(y) = \left[ w(y - by^2)^{-\lambda} + (1-w)(\log y)^{-\lambda} \right]^{\frac{1}{\lambda}}$$

and possible two parameter sub-cases are

$$U(y) = \left[ \frac{1}{2}(y - by^2)^{-\lambda} + \frac{1}{2}(\log y)^{-\lambda} \right]^{\frac{1}{\lambda}},$$

$$U(y) = w(y - by^2) + (1-w)\log y,$$

corresponding to  $\lambda = -1$  and

$$U(y) = (y - by^2)^w (\log y)^{1-w},$$

and corresponding to  $\lambda = 0$ .

The well known exponential utility  $u_4(y) = 1 - \exp(-\gamma y)$  displays CARA and hence increasing relative risk aversion (IRRA), while  $u_5(y) = \log \log y$  has  $R_R = 1 + 1/\log y$  and so has the DRRA property. The variations in combination already mentioned for  $u_1$  and  $u_2$  and  $u_3$  and  $u_2$  can be employed again although algebraic examination is rather repetitious. Combinations will show DARA but can have versatile relative risk aversion properties with regions of IRRA and DRRA depending on  $w$  and  $y$ . Besides those considered in this section, it is obvious there are many other



possible combinations of utilities.

Examples of actual employment of such combination-generated utilities do not seem to have featured in the economic literature. As mentioned already, Pratt (1964) pointed out the sum of valid utilities is a valid utility, but economists have not followed this up. The OR and management science fields provide some cases of applications, all of the  $\lambda = -1$  or sums of utilities form. Bell (1988, 1995) introduced the LINEX function,

$$U(y) = by - ce^{-\gamma y},$$

which could be seen as the sum of a linear<sup>5</sup> and an exponential. As is easily verified it displays DARA for all  $y$  and either IRRA or DRRA depending on the range of wealth. So its properties are similar to the combinations of  $u_4$  and  $u_5$  mentioned earlier. However, it is more extreme in that it is clear that for large  $y$  it effectively implies risk neutrality. But it has a ‘one-switch’ property that Bell and others feel is appropriate for a realistic utility function. Given a choice of two gambles someone with wealth  $y_1$  might prefer the ‘lower risk’ gamble, but if wealth increased to  $y_2$  could then prefer the other gamble. The ‘one-switch’ property implies further increase in wealth cannot cause any reversion to preference for the first gamble. The property may seem plausible and even innocuous, but is actually quite restrictive on the choice of utility function. As regards other utilities, Nakamura (1996) discussed the SUMEX utility function, the sum of two exponentials, although not entirely in a risk aversion context. Other papers relating to either LINEX or SUMEX include Farquhar and Nakamura (1988), Gelles and Mitchell (1999) and Bell and Fishbourne (2001). Possible explanations for the lack of applications of combination in the economic literature will be discussed in the next section.

#### IV DISCUSSION

The combination device investigated in the previous sections would appear to have considerable virtues. The device is simple enough and if we have some prior ideas about the range of risk aversion properties that should feature in any particular applied problem, we can use it to construct an appropriate utility function parsimonious in parameters. For example we might feel a power function, that is, a CRRA utility, ought to apply, but fear IRRA just might be the true situation. Then a combination of a power function and an exponential utility (which displays IRRA) would permit

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<sup>5</sup> The linear, risk neutral, function  $u(y) = by$  is concave and can be employed in (1).

testing the validity of CRRA against an alternative of IRRA and estimation of the latter if necessary. If we feared DRRA rather than IRRA, the exponential could be replaced by  $\log \log y$ . If we have other properties we also want to test – Bell’s ‘one-switch’ condition being an example – it may be possible to find a suitable combination to achieve these too.

This raises the question of why the device is not being employed. The explanation is not that flexible multi-parameter forms are currently being employed that are capable of representing the full range of risk aversion properties. There are candidate utilities with that capability, such as that of Xie (2000)<sup>6</sup> and the explicit marginal utility form of Meyer and Meyer (2005) and perhaps the HARA (hyperbolic absolute risk aversion)<sup>7</sup> form. But a flexible general purpose utility will not always be superior to one constructed to display certain features. It will depend on what we already know about a situation, or what hypothesis we particularly want to test. For example, a general purpose flexible form could nest a power form and permit testing of the adequacy of a CRRA assumption. This would be appropriate if we have no ideas on the plausibility of alternatives. The power of the test against an alternative of, say, IARA could be quite weak. But if we particularly want to choose between CRRA and IRRA we could devise a much more powerful test through an appropriate combination of simple utilities. Anyway, at least as yet, flexible multi-parameter utilities have not been frequently employed.

The simplistic explanation that researchers are just unaware of the feasibility of flexible forms or combinations will not do. There has to be a perceived problem to motivate a search for a solution and the vast majority of economic and finance literature authors seem perfectly happy with simple single parameter utilities, usually either the CARA exponential or the CRRA power (or log) form. The exponential form is very common in investment portfolio analysis probably because, assuming a normal distribution for wealth, expected utility maximisation is equivalent to familiar mean-variance analysis with a simple investor indifference curve of  $V = \mu - \gamma\sigma^2 / 2$ . But this ought not to be a compelling reason, even remaining within the context of mean-variance analysis. Meyer (1987) and Sinn (1989) have shown equivalence of expected utility maximisation and mean-variance analysis for any concave utility function and location-scale distribution. Indeed, Boyle and Conniffe (2007) have shown the equivalence can be extended to a much wider class of distributions for at least some utility functions.

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<sup>6</sup> It could be argued that Xie’s utility is deficient as regards representing IARA, but it is easily generalised (Conniffe, 2007) to a form that encompasses other utilities including IARA forms.

<sup>7</sup> The HARA utility lacks full capacity to represent DRRA.

In macro-economics, a huge volume of research, both theoretical and empirical, on the consumption function has almost always featured the power utility. It is certainly convenient for deriving results and, on occasions, it has been claimed to have empirical support. Also, various authors have sought to impose extra constraints on utility functions besides monotonicity and concavity on the grounds of increasing their behavioural ‘plausibility’. Thus Pratt and Zeckhauser (1987) defined ‘proper risk aversion’ utility functions, Kimball (1993) ‘standard risk aversion’ utilities and Caballe and Pomanski (1996) ‘mixed risk aversion’ functions. But the power function is a valid member of all classes. For example, ‘mixed risk aversion’ requires the derivatives of the utility to alternate in sign, which is true for the power function. So the power function has certainly been seen as theoretically respectable.

On the other hand, Xie (2000) has discussed the dangers implicit in assuming this CRRA utility and has argued for a more flexible form. Also, studies on the ‘equity premium puzzle’ have found it difficult, if not impossible, to reconcile observed data with a power utility function. Meyer and Meyer (2005) have argued that replacing the power utility by one permitting DRRA can provide one avenue towards a resolution of the puzzle. The paper by Roche (2006) using Xie’s function provides some further evidence on this matter. Future research in various fields may replace simple utility functions like the exponential or power by more versatile forms, or at least modify them sufficiently to permit tests of the CARA or CRRA hypotheses. Perhaps then the combination device described in this paper may have a useful role to play.

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