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Equilibrium Existence and Expected Payoffs in All-Pay Auctions with Constraints

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Abstract

This paper introduces constraints on player choices in a broad class of all-pay auctions by allowing for upper bounds on players' strategy sets. It proves the existence of equilibrium and derives simple closed-form formulae for players' expected payoffs in any equilibrium. These formulae are straightforward to calculate in applications and do not require the derivation of the equilibrium or equilibria.

This may be useful because:

(i) In some applications players' expected payoffs are the main item of interest. For example, one may be concerned about the effect of a policy on the market participants. In these cases the results can be used directly, bypassing the need for the full derivation of the equilibrium.

(ii) In all-pay auctions, equilibrium is typically in mixed strategies. So in applications where the full characterization of the equilibrium is of interest, finding the players' expected payoffs is a crucial first step in the derivation of the equilibrium.

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1. Introduction

Often agents make costly irreversible investments in hope of winning a prize. In all-pay auctions, the players with the highest scores obtain a prize each but the winners' and the losers' costs of effort are at least partially sunk. All-pay auctions are used in many areas of research including rent-seeking, political contests, lobbying, patent races, litigation, job tournaments, sports economics, advertising competition, and competition over college seats in selective universities. In any of these competitive settings, contestants may be faced with constraints.[1] For instance, in the USA a cap on political contributions restricts lobbyists who may be attempting to buy policy favors through their political donations (Che and Gale, 1998 and Pastine and Pastine, 2010). In most of Europe and in Canada politicians and political parties are faced with campaign spending limits (Meirowitz, 2008 and Pastine and Pastine, 2012b). In rent-seeking and R&D contests, participants may have liquidity constraints (Leininger, 1991). In litigation, the plaintiff and the defendant fighting over a favorable court decision have a deadline for collecting evidence (a time constraint) and they may face liquidity constraints. In the labor market, employees aiming to impress for promotion are restricted by a maximum of 24 hours of work in a day. In US professional sports leagues (NBA, NFL, NHL, MLS) teams are constrained with annual salary caps. There are score ceilings in the college admissions process as one cannot exceed 2400 on the SAT. The literature on contests with constraints has proceeded via complete characterization of equilibria in well-chosen problems. However, the need to derive the equilibrium in order to have any results naturally limits analysis to problems which are analytically tractable.

In this paper we incorporate constraints on players' actions in a broad class of complete information all-pay auctions by imposing upper bounds on the strategy sets of some, all or none of the players. We show that equilibrium exists and derive simple closed-form formulae for players' equilibrium expected payoffs.

The expected payoff formulae are straightforward to calculate and do not require the full derivation of the equilibrium or equilibria. The results are useful for two reasons: (*i*) In some applications players' expected payoffs are the main item of interest. For example, one may be concerned about the effect of a policy on the market participants. In these cases the results can be used directly, bypassing the need for the full derivation of the equilibrium. (*ii*) In all-pay auctions equilibrium is typically in mixed strategies, so in applications where the full characterization of the equilibrium is of interest, finding the players' expected payoffs is a crucial first step in the derivation of the equilibrium.

The second major result showing that equilibrium exists is non-trivial because there is a continuum of pure strategies and payoffs are discontinuous in a player's choice and so classical existence proofs do not apply. Moreover, we cannot use the innovative existence literature based on Reny (1999) as in this setting better-reply security and related concepts are generally incompatible with constraints which result in compact strategy sets. Somewhat unusually, we can guarantee equilibrium existence only for cases were players have non-compact strategy sets.

The class of contests we work from was first analyzed in Siegel (2009). The class includes standard linear all-pay auctions as well as contests with many players and multiple prizes. The framework can incorporate contests with conditional investments (costs that are paid only in victory or only in defeat), head starts, and non-ordered payoff functions. We extend this framework to include constraints on some, all, or none of the players' actions. Hence the results of Siegel (2009) are a special case.

Section 2 presents the model. The two main results are developed in Sections 3 and 4: Section 3 derives the payoff results while Section 4 presents the proof of equilibrium existence. Section 5 provides an illustrative application taken from the literature showing how the results can be used in practice. Finally, Section 6 presents a straightforward but potentially useful participation result.

2. The Model

The paper closely tracks the framework of Siegel (2009) – henceforth Siegel. Wherever possible we maintain the same notation. Here we follow the bulk of the literature by using the terminology "all-pay auction" for any contest with a perfectly discriminating contest success function. Siegel adopts a narrower definition. In order to avoid confusion when utilizing the work in Siegel we will use the terms auction and contest interchangeably, with the proviso that here "contests" will include only contests with perfectly discriminating contest success functions. In cases where we alter an assumption or a result in Siegel and the change is a strict generalization from the no-constraints case, we add "generalized" to the label of the assumption/result to make the changes clear. In cases where the assumption/result is altered and the change is not a strict generalization, we append "modified" to the label. Subsequently these qualifiers are omitted when no confusion is likely to result.

n players compete for m homogeneous prizes where $0 < m < n$. Each player i simultaneously and independently chooses a score s_i from his set of feasible scores S_i which is an interval of $\overline{\mathbb{R}}$. $a_i \in [0, \infty)$ is the initial score of contestant i if he puts forth no effort to improve his score, $a_i = \inf S_i$, and we assume that $a_i \in S_i$. Players' initial scores give their degree of headstart advantage.

The players with the highest m scores each win one prize. In the case of ties, any tie-breaking rule to allocate the prizes among the tied players is permitted. Given a profile of scores $s = (s_1, \cdots, s_n)$, player i's expected payoff is:

$$Q_i(s)\, v_i(s_i) - [\, 1 - Q_i(s)]c_i(s_i)$$

where $Q_i(s)$ is player i's probability of winning at profile s. His payoff if he wins is given by $v_i(s_i)$. His payoff if he loses is $-c_i(s_i)$. v_i and c_i are defined $\forall\ s_i \in S_i$.

There are certainly important issues that this specification cannot address. For example, it does not allow for identity-dependent externalities as analyzed in Klose and Kovenock (2015) nor does it permit non-identical prizes. However, it does allow for a broad class of all-pay auctions: It can incorporate contests with many players with potentially differing valuations, identical prizes, conditional investments, non-ordered asymmetric cost functions with players who have cost or payoff advantages in different ranges of scores, and contests with variable rewards where the value of the prize to the player depends on his own score. Note that $v_i(s_i)$ is the *net* value of winning the prize. There is no requirement that $-c_i$ be parallel to v_i. This permits analysis of contests where players make conditional investments. For example, an Olympic committee may promise to build a stadium if the games are held in their city. By specifying a range of s_i in which c_i is constant while v_i is decreasing, the framework permits such promises of actions to be taken only in victory. Disconnecting the cost of losing and the value of winning also permits analysis of situations where the difficulty in carrying out an action depends on the outcome of the competition. For example, a politician who borrows money for his campaign may find it easier to raise the funds to repay the loan if he is elected.

Denote $k_i = \sup S_i$ so that for cases where $k_i < \infty$, k_i is a ceiling on the player's choice of score. Such an upper bound on a player's strategy set will be termed a constraint. Section 5 presents an example showing how constraints on players' choices result in upper bounds on players' strategy sets. We permit either $k_i \in S_i$ or $k_i \notin S_i$. The introduction of a constraint is without loss of generality as the affinely extended real numbers permit the notation $k_i = \infty$ to represent the absence of a constraint. Constraints are permitted for any, none or all players and at any scores. Hence the paper generalizes Siegel in which players do not have constraints, *i.e.* $k_i = \infty$ for all players.

To proceed we need to place three assumptions on v_i and c_i:

Assumptions:

A1: v_i *and* $-c_i$ *are continuous and nonincreasing*

Generalized A2: $c_i(a_i) = 0$, $v_i(a_i) > 0$ *and if* $k_i = \infty$ *then* $\lim_{s_i \to \infty} v_i(s_i) < 0$

A3: $c_i(s_i) > 0$ *if* $v_i(s_i) = 0$

The framework permits analysis of auctions where effort increases the value of the prize. However, the assumption on v_i in A1 implies that conditional on winning an increase in the score does not increase the value of the prize by more than the cost of additional effort. A3 and the assumption on c_i in A1 capture the feature of all-pay auctions where the winners' and the losers' costs of effort are at least partially sunk. However, they do not require that the cost of incremental effort is sunk. The A2 implies that a prize has a strictly positive value for each player and the payoff conditional on winning is negative with a high enough score.

It is useful to define some terminology which is key to the analysis.

Definitions:

(i) A player i is said to be *restricted at* x if $x = k_i$ and one of two conditions are satisfied: (a) $x \in S_i$ and $v_i(x) > 0$ or (b) $x \notin S_i$ and $\lim_{z \to x^-} v_i(z) > 0$. So, a player is restricted at x if he has a positive value from winning at score at x or approaching x from below, but he is unable to exceed that score due to his constraint.

(ii) Player i's generalized *reach*, r_i, is the supremum of the feasible scores at which the player's valuation for winning is non-negative, $r_i = \sup\{s_i \in S_i : v_i(s_i) \geq 0\}$. Re-index players in any decreasing order of their reach, so that $r_1 \geq \cdots \geq r_m \geq \cdots \geq r_n$.

(iii) Player $m+1$ is the *marginal player*. The indexing of the players ensures that there is only one marginal player. If there are multiple players with the same reach the identity of the marginal player will be arbitrary but there will only be a single marginal player.

(iv) The *threshold*, T, of the contest is the reach of the marginal player: $T = r_{m+1}$.

(v) Player i's generalized *power*, w_i, is his valuation of winning at his highest feasible score that is less than or equal to the threshold if he is able to choose such a score. If he can only choose scores above the threshold, his power is his valuation from winning at his lowest feasible score. Formally, if $a_i \leq T$ let $z = \sup\{s_i \in S_i : s_i \leq T\}$ and if $a_i > T$ let $z = a_i$. Player i's power is:

$$w_i = \begin{cases} v_i(z) & \text{if } z \in S_i \\ \lim_{x \to z^-} v_i(x) & \text{if } z \notin S_i \end{cases}$$

The definitions of *reach* and *power* are generalizations of the concepts from Siegel to permit the possibility that players may be constrained at their reach.

When players have no constraints, players with $r_i \leq T$ have power $w_i \leq 0$. However with constraints, the power of players with $r_i \leq T$ may be positive. If a player $i > m$ is constrained at his reach, then it is possible that $w_i > 0$. For instance, consider the contest in Figure 1 with one prize and two players. Player 1 has no constraint. Player 2's valuation of the prize is high but he is constrained and cannot achieve a score greater than k_2. The reach of Player 1 is r_1. The reach of Player 2 is k_2. Since $r_1 > r_2$, Player 2 is the marginal player. The threshold of the contest is $T = k_2$. The marginal player has a higher power than player 1, $w_2 > w_1 > 0$, which cannot happen in the model without constraints.

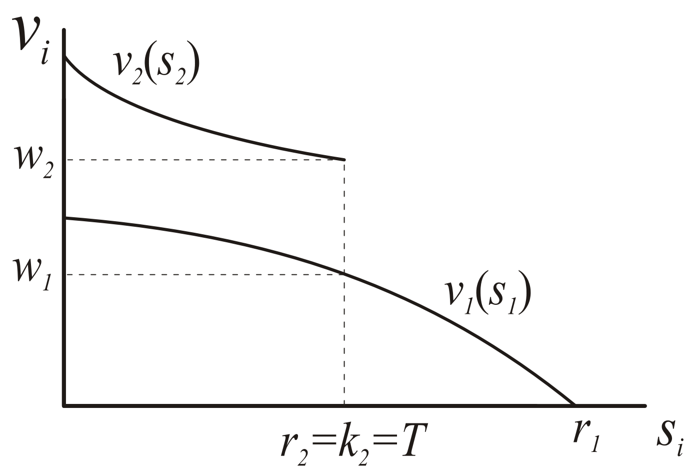

Figure 1

In order to focus on the pertinent issues, in the figures for the examples we graph only v_i and not c_i. Notice that given $v_i(a_i) > c_i(a_i) = 0$, the definitions of reach, threshold and power are based only on the players' valuations from winning and not on the shape of players' costs of losing, $c_i(s_i)$. Under the assumptions above and some additional assumptions below, it will be shown that to find player i's expected payoff in any equilibrium, all one needs to calculate is the players' reaches, the threshold and w_i. While a player's cost of losing and shape of v_i away from the threshold typically alter the equilibrium, they do not affect the expected payoffs.

We need to place additional restrictions on the game, termed *generic* conditions. These conditions restrict attention to games that are generic in the sense that any auction that fails to meet the conditions can be perturbed slightly so that it does. In many sufficiently parameterized models the generic conditions will hold with probability one if the relevant parameters are drawn from continuous distributions before they become common knowledge. For example, the applications presented in Section 5 have this feature.

The generic conditions are needed to guarantee that there is at least one equilibrium where tied scores occur in with probability zero - although such ties are permissible. When the generic conditions do not hold, general statements about player expected payoffs are not likely to be forthcoming. Klose and Kovenock (2015) show that such games can have multiple equilibria which are not payoff equivalent.

The generic conditions are divided into two groups, *weak generic conditions* and *strict generic conditions*. A contest that satisfies the weak generic conditions, is a *weakly generic contest.* The weak generic conditions are requirements on the game in the neighborhood of the threshold. They are sufficient to establish the expected payoff result. If an auction satisfies both the weak and strict generic conditions, it is a *strictly generic contest*. The strict generic conditions are requirements on the game in the neighborhood of the constraints of players whose constraints may be binding in equilibrium. Any auction without constraints that is generic in the sense of Siegel satisfies both sets of generic conditions and hence any generic contest without constraints is also a strictly generic contest.

Section 3's results deriving simple closed-form formulae for players' expected payoffs in any equilibrium are valid for contests that satisfy the following weak generic conditions:

(i) Generalized Power Condition: The marginal player is the only player with reach at the threshold and players $\{1, \cdots, m\}$ *have non-zero power.*

(ii) Generalized Cost Condition: The marginal player has strictly decreasing valuation from winning just below the threshold. That is, for any $x \in [a_{m+1}, T)$, $v_{m+1}(x) > lim_{z \to T^-} v_{m+1}(z)$.

The Generalized Power Condition parallels Siegel's Power Condition which requires that the marginal player is the only player with power zero. However with constraints the marginal player may be restricted at the threshold. As in Figure 1, there may be no player with zero power. It is also possible that a player $i > m+1$ has power zero if $v_i(k_i) = 0$. The conditions are equivalent in cases where $k_i > r_i \ \forall \ i > m$.

Note that the Generalized Power Condition rules out the cases where the value of winning v_i of any player $i \in \{1, \cdots, m\}$ is zero at the threshold. The Generalized Cost Condition rules out cases where the value of winning of the marginal player, v_{m+1}, is constant in s_i in the neighborhood just below the threshold. Contests that do not meet the weak generic conditions can be perturbed slightly to meet them. For instance, if there are two players with the same reach at the threshold, giving one of the players an arbitrarily small headstart advantage or the slightest valuation advantage can create a contest that meets the Generalized Power Condition. Likewise, perturbing the marginal player's head start or valuation for winning around the threshold can generate a contest that meets the Generalized Cost Condition.

At first glance, the Generalized Power Condition may seem problematic in applications where constraints are legal prohibitions. It seems intuitive that there may be many players with reach at the threshold since all players may be subject to the same legal constraint. However, legal restrictions are typically on actions rather than on scores. Arbitrarily small differences in players' head starts or technology of converting actions into scores will lead to different score constraints even with identical constraints on actions. For instance, in Che and Gale (1998) political donors face the same contribution cap. The contest does not satisfy the Generalized Power Condition because the donors have the identical effectiveness in converting donations into political influence (scores). However, the political donation contest satisfies the Generalized Power Condition if the donors have even the slightest difference in technology of converting donations into influence, see Pastine and Pastine (2010). Likewise, the Generalized Power Condition is satisfied if the politician has any initial policy preference - however small - providing one of the contributors a head start.

It will prove useful to define several sub-groups of players. Let $N_w = \{1, \cdots, m\}$ denote the set of players with the m highest reaches. In any weakly generic contest each player in N_w has reach greater than the threshold. $N_L = \{m+1, \cdots, n\}$ denotes the set of remaining players. All players in N_L have reaches less than or equal to the threshold. Define $\hat{N}_L$ as the subset of players in N_L who have reaches equal to their constraint, $\hat{N}_L = \{i \in N_L : r_i = k_i\}$.

Since for the players in N_L scores $s_i > T$ are either infeasible or strictly dominated by a_i, the players in N_L are either unable or unwilling to exceed the threshold. So any of the m players in N_w can guarantee victory by choosing $s_i = \max\{a_i, \ T + \varepsilon\}$ for sufficiently small, positive ε. They do not have to go all the way up to their reach in order to ensure victory. So in equilibrium their constraints will not be binding. Players in $N_L \setminus \hat{N}_L$ have reaches less than their constraint. In equilibrium their constraints will not be binding. Therefore $\hat{N}_L$ contains all the players whose constraints may possibly be binding in equilibrium.

In order to guarantee that an equilibrium exists we need to impose additional generic conditions in the neighborhood of the constraints of the players in $\hat{N}_L$. A contest is strictly generic if in addition to the above weak generic conditions, it satisfies the following strict generic conditions:

(i) *Strategy Set Condition:* No player in $\hat{N}_L$ has $s_i = k_i$ in his set of possible choices. That is $k_i \notin S_i \, \forall \ i \in \hat{N}_L$.

(ii) *Strict Cost Condition:* All players in $\hat{N}_L$ have their valuation from winning strictly decreasing just below their reach and if player $m+1$ is in $\hat{N}_L$ then he has a strictly positive payoff from winning for scores approaching his constraint. That is $\forall\ i \in \hat{N}_L$, $v_i(x) > \lim_{z \to r_i^-} v_i(z)$ for any $x \in [a_i,\ r_i)$ and if $m+1 \in \hat{N}_L$ then $\lim_{z \to k_{m+1}^-} v_{m+1}(z) > 0$.

(iii) *All-Pay Condition:* All players in $\hat{N}_L$ have a positive cost of losing when approaching their constraint. That is $\forall\ i \in \hat{N}_L$, $\lim_{z \to k_i^-} c_i(z) > 0$.

(iv) *Constraint Condition:* No two players in $\hat{N}_L$ have identical constraints. That is $\forall\ i,\ j \in \hat{N}_L$ where $i \neq j$, $k_i \neq k_j$.

Since these strict generic conditions are on the specification of the game in the neighborhood of the constraints of players in $\hat{N}_L$, any weakly generic contest without constraints is also strictly generic. These are generic conditions in the sense that any contest that fails to meet them can be perturbed slightly to meet them.

The Strategy Set Condition permits constraints of the form "spending must be less than x" but precludes constraints of the form "spending cannot be greater than x." Any contest that fails to meet the Strategy Set Condition can be perturbed to meet it by removing a single point (k_i) from the set of possible choices for each of the offending players. Also notice that this condition implies $k_i > a_i$ for players in $\hat{N}_L$. It is interesting to note that many games in the literature involve spending money to increase players' scores. In these cases the underlying reality is a discrete-choice game as monetary units are not infinitely divisible. Continuous-choice games are analyzed solely due to their tractability. However in these cases, the choice of open or closed intervals for constraints is entirely arbitrary. In reality "spending cannot be greater than \$10,000" is equivalent to "spending must be strictly less than \$10,000.01." When moving to a continuous-choice approximation of the discrete-choice reality, there is no reason to prefer one over the other except for tractability.

The first part of the Strict Cost Condition is an extension of the Generalized Cost Condition to players in $\hat{N}_L$, rather than applying it just to the marginal player. The second part says that if the marginal player's constraint may be binding he has a positive payoff from winning approaching his constraint. As such, any game that fails to meet the Strict Cost Condition can be perturbed to meet it by increasing any offender's payoff from winning at or just below the constraint by an arbitrarily small amount.

The All-Pay Condition requires that for players in $\hat{N}_L$ at least some of the cost of effort is sunk locally in the neighborhood of their constraint. The contest has an all-pay nature for those players. A contest that fails to meet the All-Pay Condition can be perturbed to meet it by adding an arbitrarily small amount to each offending player's cost of losing just below his constraint.

The Constraint Condition guarantees that no two players have binding constraints at the same score. A contest that fails to meet the Constraint Condition can be perturbed to meet it by arbitrarily small changes to the offending players' constraints. Again at first glance this may seem problematic in applications where the constraints come from legal prohibitions, since all players may be subject to the same laws. However, arbitrarily small differences in players' head starts or technology of converting actions into scores will lead to different score constraints even with identical constraints on actions. The application in Section 5 provides an example from the literature which illustrates this.

3. Payoff Characterization

In this section we develop the characterization for the expected payoffs in any equilibrium of any weakly generic contest. Three lemmas are used in the payoff characterization. The first two of these intermediate steps are modifications of the corresponding items in Siegel rather than strict generalizations. Here we permit constraints but confine the domain to weakly generic contests whereas the proofs in Siegel apply to any unconstrained contest whether generic or not.

For each player define G_i as a cumulative distribution function that assigns probability one to scores in S_i and let $G_i(s_i)$ be that c.d.f. evaluated at score s_i. For a strategy profile $G = (G_1, \cdots, G_n)$, let $G_{-i} = (G_1, \cdots, G_{i-1}, G_{i+1}, \cdots, G_n)$, the strategy profile of all players except player *i*. $P_i(s_i; G_{-i})$ is player *i*'s probability of winning when he chooses $s_i \in S_i$ and all other players play according to G. Similarly define expected utility $u_i(s_i; G_{-i}) = P_i(s_i; G_{-i})\, v_i(s_i) - [1 - P_i(s_i; G_{-i})]c_i(s_i)$.[2] The score s_i is in player *i*'s best response set if $s_i \in \arg\max_{x \in S_i} u_i(x; G_{-i})$. s_i is in the support of player *i*'s strategy if it is chosen with non-zero probability in G_i. G forms an equilibrium if in G for each player *i*, all scores in the support of *i*'s strategy are in his best response set.

Modified Least Lemma: *In any equilibrium of a weakly generic contest, the expected payoff of each player in N_w is at least his power and the expected payoff of each player in N_L is at least zero .*

Proof: In equilibrium no player would choose a score higher than his reach since such a score is either infeasible or is strictly dominated by a_i. By the definition of a player's power and the threshold at most m players can have reach strictly greater than T. Since players $i \in N_w$ who have $a_i \le T$ are not restricted at T and are able to exceed the threshold by ε (Assumption A1), they can guarantee an expected payoff that is equal to their power. Players $i \in N_w$ who have $a_i > T$ can win with certainty at $s_i = a_i$ by the Power Condition and hence can guarantee a payoff equal to their power. Each player $i \in N_L$ can guarantee a payoff of at least zero by simply choosing a_i. *Q.E.D.*

The Modified Least Lemma establishes a lower bound for player payoffs in any equilibrium. We now need to establish an analogue of the Zero Lemma (Siegel pg 80) showing that for the players in N_L the expected payoff must be equal to this lower bound. In Siegel this is done using an intermediate step termed the Tie Lemma (Siegel pg 80) which shows that in any equilibrium if two or more players play strategies with atoms at the same score (choose that score with strictly positive probability) then all such players either win or lose with certainty at that score. In the next section we will develop an analogue of the Tie Lemma for strictly generic contests with constraints. As in the original Tie Lemma, the proof relies on a player's ability to increase his score slightly to avoid ties when his probability of winning is positive but less than one in a tie. Unfortunately, this is not always possible for ties at k_i when $k_i \in S_i$. However, we will need to show the expected payoff result for cases where $k_i \in S_i$ in order to prove the existence of equilibrium in the next section. Therefore here we tweak the proof of the Zero Lemma to bypass the use of the Tie Lemma:

Modified Zero Lemma: *In any equilibrium G of a weakly generic contest, all players in N_L must have scores in the support of their strategies in G with which they win with probability zero or arbitrarily close to zero. These players have expected payoff of zero.*

Proof: Let J denote a set of players including the m players in N_w plus any one other player $j \in N_L$. Let $\tilde{S}$ be the union of the best-response sets of the players in J and let $s_{\inf}$ be the infimum of $\tilde{S}$. Consider three cases: (*i*) two or more players in J have an atom at $s_{\inf}$, (*ii*) exactly one player in J has an atom at $s_{\inf}$, and (*iii*) no players in J have an atom at $s_{\inf}$. Examination of these cases helps establish the expected payoffs of players in N_L.

Case (i). Initially denote $N' \subseteq J$ as the set of all players in J with an atom at $s_{\inf}$ where $|N'| > 1$. Every player in $J \setminus N'$ chooses scores greater than $s_{\inf}$ with probability 1. Therefore even if every player that is not in J chooses scores strictly below $s_{\inf}$ with probability 1, that leaves one too few prizes to be divided between $|N'|$ players. So not all players in N' can win at $s_{\inf}$ with certainty.

If there are any players in N' with probability of winning at $s_{\inf}$ equal to 1, remove them from N' so that $P_i(s_{\inf}; G_{-i}) < 1 \ \forall\ i \in N'$. If $|N'|=1$ then that player i loses with certainty with score $s_{\inf}$ and i's expected payoff cannot be positive. From the Modified Least Lemma and the Generalized Power Condition this player cannot be in N_w, so he must be the one player in $J \setminus N_w$, and he must have expected payoff equal to zero. If $|N'| > 1$, then let H be the set $N' \cap N_w$. Since there is only one player in $J \setminus N_w$, $|H| \in \{|N'|-1, \ |N'|\}$. For no player $i \in H$ can the probability of winning at $s_{\inf}$ be equal to zero. If it were, i would have $u_i(s_{\inf}; G_{-i}) \leq 0$ and he must have a positive payoff by the Modified Least Lemma and the Generalized Power Condition because $H \subset N_w$. If player i loses ties with other players in N' with positive probability, $P_i(s_{\inf}; G_{-i}) \in (0,1)$. But this is not possible for any $i \in H$, since i can do better by increasing his score slightly above $s_{\inf}$ to avoid ties by the Generalized Power Condition. Hence at $s_{\inf}$ every player in H must win every tie with other players in N'. This is not possible if $|H| = |N'|$ since there are not enough prizes for all the players in N'. Hence $|H| = |N'|-1$ so $j \in N'$ and j loses all ties with members of N' at $s_{\inf}$. Therefore $P_j(s_{\inf}; G_{-j}) = 0$ and $u_i(s_{\inf}; G_{-i}) \leq 0$ since $j \in N'$ and $j \in N_L$. By the Modified Least Lemma his expected payoff must be zero.

Cases (ii) and (iii). The corresponding proofs in Siegel apply without modification and establish that in both cases one player $i \in J$ has a score in the support of his strategy in which he wins with probability 0 or arbitrarily close to 0 and has an expected payoff of at most 0. By the Modified Least Lemma i must have a payoff of 0, and by the Generalized Power Condition $i \in N_L$ and so $i = j$.

The above applies for each player $j \in N_L$. *Q.E.D.*

Generalized Threshold Lemma: *In any equilibrium G of a weakly generic contest, the players in N_w have scores in the support of their strategies in G that approach or exceed the threshold and, therefore, the players in N_w have an expected payoff of at most their power.*

The proof is omitted here as the proof of the Threshold Lemma in Siegel applies without modification noting only that with constraints players in $N_L \setminus \{m+1\}$ may or may not have negative powers, however they still have reaches strictly below the threshold.

From these intermediate results we can establish the first of the two main results of the paper. The Expected Payoff Result is a generalization of Theorem 1 in Siegel.

Expected Payoff Result: *In any equilibrium of a weakly generic contest, the expected payoff of each player in N_w is equal to his power which is greater than zero, and the expected payoff of each player in N_L is zero, which is less than his power if he is restricted at his reach.*

Proof: The Modified Least Lemma and the Generalized Threshold Lemma establish that players in N_w have expected payoffs equal to their power which is greater than zero by the Generalized Power Condition. The Modified Zero Lemma establishes that the players in N_L have expected payoffs equal to 0. If a player in N_L is not restricted at his reach, his power is less than or equal to zero. If he is restricted at his reach his power is greater than zero so his expected payoff is less than his power. *Q.E.D.*

Because players' expected payoffs from the contest depend only on the order of their reaches and on their valuation of winning at the threshold, the striking implication of Siegel continues to hold in contests with constraints; The players' costs of losing and the shape of v_i away from the threshold do not affect equilibrium expected payoffs. They will typically have an effect on equilibrium strategies, but not on expected payoffs. Similarly, presuming that an equilibrium still exists (which will be addressed in the next section) and that the contest remains weakly generic, a change in the constraint of any player other than the marginal player does not affect the expected payoff of any player as long as the change does not alter the identity of the marginal player:

Implications of the Expected Payoff Result: *In any equilibrium of a weakly generic contest, consider a small change in a player's constraint such that the identity of the marginal player remains the same:*

▪ *A small change in the constraint of any player other than the marginal player does not affect the payoff of any player.*

▪ *A small change in the marginal player's constraint does not affect the payoff of any player in N_L (including his own).*

▪ *If the marginal player is restricted at k_{m+1} then relaxing his constraint (weakly) decreases the expected payoff of each player in N_w.*

Proof: Follows directly from the Expected Payoff Result, assumption A1, the definitions of reach and power and the Generalized Cost Condition. By the definitions of reach and power any change to a single player's constraint that doesn't alter the identity of the marginal player will leave the members of the sets N_L and N_w unaltered. Since there is only one player with reach at the threshold by the Generalized Cost Condition, a change in one player's constraint that doesn't alter the identity of the marginal player can only alter the threshold if that player with the changed constraint is the marginal player. If the player with the changed constraint is not the marginal player then all players in N_L have expected payoff of zero before and after the change and all players $i \in N_w$ have expected payoff of $v_i(T)$ both before and after the change, proving the first part. Since the members of N_L are unaltered and include the marginal player, by the Expected Payoff Result they all have an expected payoff of zero before and after the change in constraint, proving the second part. By the definitions of reach and the threshold, if the marginal player is restricted at k_{m+1}, then relaxing his constraint (increasing k_{m+1}) will increase the threshold. By the Expected Payoff Result players $i \in N_w$ have expected payoff of $v_i(T)$ so by A1 increasing T weakly decreases $v_i(T)$, proving the third part. *Q.E.D.*

It follows that a player's expected payoff is affected by a change in his own constraint only if the change in his constraint switches him between N_w and N_L. Other changes in his constraint will typically affect equilibrium strategies, but they will not affect the player's own payoffs.

This may have useful applications. Consider a two-stage game in which in the first stage players invest in relaxing their constraints by increasing their credit limits, adding factory capacity, registering voters, building an R&D lab etc. Then in the second stage they engage in a contest. The Expected Payoff Result in the second stage implies that the investment decision in the first stage would also have an all-pay contest structure. Players would either want to invest enough to become player *m*, and not more, or they would not want to invest at all.

4. Existence of Equilibrium

In this section we derive the second main result of the paper which shows that a Nash equilibrium exists in any strictly generic contest. Classical existence proofs do not apply because players have a continuum of pure strategies and their payoffs are discontinuous in their choices. A literature based on the path-breaking work of Reny (1999) has made great strides in proving existence in discontinuous games. While this literature can be used to show that equilibrium exists in unconstrained all-pay contests, unfortunately it does not apply to all-pay contests with constraints.

For example, Monteiro and Page (2007) shows that any compact game that is upper semicontinuous and uniformly payoff secure has an equilibrium in mixed strategies. The all-pay auction without constraints has these features, but the same auction with constraints does not. Uniform payoff security means that in any strategy profile each player has a strategy he can use to guarantee almost the same payoff if other players make small deviations from their strategies. In a contest where all players' constraints are higher than their reaches (an effectively unconstrained contest) if other players make small changes to their strategies each player can guarantee at least epsilon below his current payoff by increasing his score slightly or, if that is higher than his reach, by choosing a_i. Hence such contests are uniformly payoff secure. However, with constraints this is not the case. Epsilon above a players' current choice may not be feasible while still providing a positive expected payoff at the initial strategy profile. This problem can be corrected by making $k_i \notin S_i$ for each player. In that case for any $s_i \in S_i$, $s_i + \varepsilon \in S_i$ for sufficiently small, positive ε. However in this case players' strategy sets are no longer compact. Hence, in order to satisfy the payoff-security condition we must violate the compact strategy set condition. Similar issues arise when trying to apply any of the existence proofs based on variations of better-reply security. Hence we have to proceed in another way.

Most existence proofs require compact strategy sets while here we impose the Strategy Set Condition which requires non-compact strategy sets for a subset of the players and permits them for all players. This is to deal with a specific problem that arises in finding equilibria in constrained contests. The issue is best illustrated by a series of examples which provide insight into the approach used in the existence proof.

Example 1: Consider a standard linear two-player all-pay auction with a single prize with a common value equal to two, $v_i(s_i) = 2 - s_i$ and $c_i(s_i) = s_i$. Player 1's constraint is higher than the value of the prize, $S_1 = [0, 3]$, but player 2 is constrained at one, $S_2 = [0, 1)$. Ties are decided by coin flip. This contest is strictly generic. Player 2 has a reach of one, player 1 has a reach of 2 and hence player 2 is the marginal player and the threshold is one. Equilibrium exists and is in mixed strategies which are given by the following cumulative distribution functions: $G_1(s_1) = s_1/2$ for $s_1 \in [0, 1)$, and $G_1(s_1) = 1$ for $s_1 \geq 1$. $G_2(s_2) = \frac{1}{2} + s_2/2$ for $s_2 \in [0, 1)$. So player 1 puts an atom of probability of 1/2 at $s_1 = 1$ and spreads the remaining probability uniformly over $[0, 1)$. Player 2 puts an atom of 1/2 at $s_2 = 0$ and spreads the remaining probability uniformly over $(0, 1)$. This gives an expected payoff equal to one for player 1 and an expected payoff of zero for player 2 as required by the Expected Payoff Result and no player has a profitable deviation.

Example 2: Take the contest from Example 1 and change the action space for player 2 to include his constraint, $S_2 = [0, 1]$. In this case the contest is weakly generic but it is not strictly generic as it violates the Strategy Set Condition. It is now possible for player 2 to deviate from the equilibrium strategy in Example 1, matching player 1 at his atom at $s_1 = 1$. Deviating to $s_2 = 1$ gives player 2 a probability of winning of 3/4 and an expected payoff of 1/2. This is a violation of the Expected Payoff Result since player 2 is still the marginal player and therefore must have an expected payoff of zero. In order to create an analogue of the equilibrium in Example 1, player 1 needs to move his probability mass high enough so that player 2 cannot match that choice. So he needs to move it just above $s_1 = 1$. But "just above" is not defined.

Example 3: Take the contest of Example 2 but abandon the tie-breaking rule. Replace it with a rule in which ties are decided by coin flip everywhere except at $s_i = 1$ where all ties are decided in favor of player 1. This may or may not be a reasonable tie-breaking rule in a particular application, but it does mean that the equilibrium strategies from Example 1 also form an equilibrium in Example 3. It is not possible for player 2 to capture an excessive payoff by deviating to $s_2 = 1$, even though that score is now technically feasible.

In general, when player i is restricted at his constraint, which may be the case for any of the players in $\hat{N}_L$ not just the marginal player, equilibrium may require a rival to put an atom just above i's highest feasible score. This is well defined only when the Strategy Set Condition holds. However, the tools in the literature for proving existence of equilibrium largely require compact strategy sets. Hence we start with a game like Example 1, except that we don't yet know whether equilibrium exists. Then we create a new game with compact strategy sets as was done moving from Example 1 to Example 2. Then we use the results of Simon and Zame (1990) to show that there exists some tie-breaking rule under which equilibrium exists in the new game, as in Example 3. We then establish that at least one of the equilibria of the new game with the new tie-breaking rule is also an equilibrium of the original game with its tie-breaking rule replaced by the new rule. And finally we use an insight from Siegel to show that this equilibrium is also an equilibrium of the original game for any tie-breaking rule. It is important to note that while the result of Simon and Zame (1990) is used as an important intermediate step, it is just an intermediate step. The existence result applies to contests with any tie-breaking rule, not just the special rule from Simon and Zame (1990).

We will first need to establish the Tie Lemma, which in the unconstrained case held for all contests whether generic or not. The proof is built on the ability of all players with atoms at x to exceed x if desired, and so the lemma does not apply for contests that are not strictly generic as a player cannot exceed x if $x = k_i$ which can happen when the Strategy Set Condition does not hold.

Modified Tie Lemma: *In any equilibrium of a strictly generic contest, if two or more players have an atom at a score x, that is, choose x with a strictly positive probability, then players who have an atom at x either all win with certainty or all lose with certainty when choosing x.*

Proof: Since their reaches are less than or equal to the threshold none of the $(n-m)$ players in N_L will choose scores exceeding the threshold. Hence if $x > T$ then any player choosing x will win with certainty, satisfying the lemma. So we only need to consider $x \leq T$. By the Generalized Power Condition all players in N_w have reaches greater than T so $x < k_i \ \forall\, i \in N_w$. By the definition of $\hat{N}_L$ all players in $N_L \setminus \hat{N}_L$ have $r_i < k_i$. These players will only place an atom at x if $x \leq r_i$ and so if any player $i \in N_L \setminus \hat{N}_L$ places an atom at x it must be the case that $x < k_i$. Finally, if any player $i \in \hat{N}_L$ places an atom at x it must be the case that $x < k_i$ since $k_i \notin S_i$ by the Strategy Set Condition. Thus for any players placing an atom at $x \leq T$ it must be that $x < k_i$. Hence all players with an atom at $x \leq T$ have $x + \varepsilon \in S_i$ for sufficiently small $\varepsilon > 0$. From this, the proof of the Tie Lemma in Siegel applies without modification and hence is omitted here. *Q.E.D.*

We now prove the second primary result of the paper. This is a modification of the existence result in Siegel, Corollary 1. It is not a strict generalization because Siegel shows the existence of equilibrium for any unconstrained contest, whether generic or not. Here we include contests with constraints but as a consequence have to limit the domain to strictly generic contests.

Existence of Equilibrium Result: *Every strictly generic contest has a Nash equilibrium.*

Proof: Take a strictly generic contest and define N_a as the set of players who are constrained at their head start, $N_a = \{i \in N : a_i = k_i\}$. By the Generalized Power Condition, the definition of $\hat{N}_L$ and the Strategy Set Condition, $N_a \subseteq N_w$ and hence $|N_a| \leq m$ and $a_i > T\ \forall\ i \in N_a$. If $|N_a| = m$ then m players have head starts which exceed the threshold so each player $i \in N$ playing a pure strategy of $s_i = a_i$ is an equilibrium, and the result holds. If $|N_a| < m$ construct a new contest C which is identical to the original contest except with the players in N_a and $|N_a|$ prizes removed. In the original game the players in N_a are by necessity entirely passive since for them $S_i = \{a_i\}$ and they each win a prize with certainty since $a_i > T$. Hence removing them and their prizes does not change the strategic environment for any remaining player at any feasible score that is not strictly dominated. The contest is still strictly generic, the threshold does not change, players in N_L are either unwilling or unable to exceed the threshold and players in N_w win with certainty with any score greater than the threshold in both games. Hence for the players that are in both games, the strategies that form an equilibrium in C will also form an equilibrium in the original game, with the players in N_a playing their only feasible strategy of choosing $s_i = a_i$ with certainty. Hence it suffices to show that C has an equilibrium. In what follows define all variables with respect to contest C. So N is the set of players in C, m is the number of prizes in C, players are indexed with respect to the order of their reaches in C and so on.

Consider a new contest C' which is identical to C except that each player's set of feasible scores is capped at $K = \max_{i \in N} r_i < \infty$. So for each player i in C', $S_i' = S_i \bigcap [a_i, K]$. Define each player's constraint in C' as $k_i' = \sup S_i' = \min\{k_i, K\}$. Since in C scores greater than K are either infeasible or strictly dominated by a_i, any equilibrium of C' is also an equilibrium of C. Hence it is sufficient to show that C' has an equilibrium.

Let N_k be the set of players whose constraints are not in their strategy set in C', $N_k = \{i \in N : k_i' \notin S_i'\}$. Create a third contest C'' which is identical to C' but with an expanded action space for the players in N_k so that all players in C'' have k_i' in their strategy set. Specifically, for each $i \in N$ let $S_i'' = [a_i, k_i']$, for all $s_i \in S_i'$ let $v_i''(s_i) = v_i(s_i)$ and $c_i''(s_i) = c_i(s_i)$, and for all $\{s_i \in S_i'' : s_i \notin S_i'\}$ let $v_i''(s_i) = \lim_{z \to k_i'^-} v_i(z)$ and $c_i''(s_i) = \lim_{z \to k_i'^-} c_i(z)$, noting that $\{s_i \in S_i'' : s_i \notin S_i'\}$ only for $i \in N_k$ at $s_i = k_i'$. So we have added a single point, k_i', to the strategy set of players in N_k ensuring that all players have compact strategy sets. The resulting game is still weakly generic, but it is not necessarily strictly generic. It continues to meet all the other conditions but it violates the Strategy Set Condition for players in $\hat{N}_L$. However, because C'' has compact strategy sets it is more amenable to analysis than C'.

In particular, the results of Simon and Zame (1990) show that if we abandon the tie-breaking rule shared by C' and C'', then there exists some tie-breaking rule, which may be dependent on the score and/or identity of the players, in which C'' has at least one mixed-strategy equilibrium when that tie-breaking rule is employed. Denote the games when this tie-breaking rule is employed by $\tilde{C}'$ and $\tilde{C}''$ respectively. So $\tilde{C}''$ has at least one equilibrium, but strategy profiles that form an equilibrium in $\tilde{C}''$ may or may not form an equilibrium in $\tilde{C}'$. However below we show that there is at least one equilibrium of $\tilde{C}''$ whose equilibrium strategies also form an equilibrium in $\tilde{C}'$. The key to this is showing that there is at least one equilibrium of $\tilde{C}''$ in which no player puts an atom at his constraint, k_i'.

Take an equilibrium G of $\tilde{C}''$ and a player $i \in N_w$. Take a small $\varepsilon > 0$ and let $b = \max\{a_i, T + \varepsilon\}$. Since i can win with certainty with any $s_i > T$ by the Generalized Power Condition, if there exists an $\varepsilon > 0$ such that $v_i''(k_i') < v_i''(b)$ then in G_i he must place zero probability on $s_i = k_i'$. By A1 the only other possibility is $v_i''(k_i') = v_i''(b)\ \forall\ \varepsilon > 0$, however small. In this case it is possible that in G_i player i has an atom at $s_i = k_i'$. Suppose this is the case and construct an alternative strategy for i which is the same as G_i but with the upper end of the distribution truncated at some score $h \in [b, k_i')$: $\hat{G}_i(s_i) = G_i(s_i)\ \forall\ s_i < h$ and $\hat{G}_i(s_i) = 1\ \forall\ s_i \geq h$. Replace i's strategy in G with this new strategy. G still forms an equilibrium of $\tilde{C}''$. Player i wins with certainty with all $s_i > T$ and each score gives the same payoff so altering his strategy does not alter his own payoffs. The new strategy for i does not alter the probability of winning for any player at any score less than h, so such scores still yield the same expected payoff for each player. For all $j \in N_L$, scores $s_j \geq h$ are either infeasible or strictly dominated by a_j, so G_j still forms a best response. For all $j \in N_w$, player j wins with certainty with any $s_j > T$ whether i plays G_i or $\hat{G}_i$, so the new strategy does not change the expected payoff of j for any s_j. By iterating this argument over each $i \in N_w$ we can construct an equilibrium G of $\tilde{C}''$ in which no player in N_w places an atom at $s_i = k_i'$. In what follows consider such an equilibrium.

We now show that in G no player in N_L places an atom at his constraint either. The reach of each player in $i \in N_L \backslash \hat{N}_L$ is strictly less than his constraint and so $s_i = a_i$ strictly dominates $s_i = k_i'$ by A3. So we only need to consider the players in $\hat{N}_L$. The proof will proceed by contradiction:

Suppose that there exists an $i \in \hat{N}_L$ with an atom at $s_i = k_i'$. Considering i's probability of winning when choosing $s_i = k_i'$, one of three cases must be true: $P_i(k_i'; G_{-i}) = 0$, $P_i(k_i'; G_{-i}) = 1$ or $P_i(k_i'; G_{-i}) \in (0, 1)$.

Case 1: $P_i(k_i'; G_{-i}) = 0$. By the All-Pay Condition player $i \in \hat{N}_L$ receives a negative payoff when choosing $s_i = k_i'$, a contradiction of the Expected Payoff Result.

Case 2: $P_i(k_i'; G_{-i}) = 1$. If $i>m+1$ then $k_i' < r_{m+1}$ by the Generalized Power Condition, so player $m+1$ can choose $s_{m+1} = \max\{a_{m+1}, k_i' + \varepsilon\}$ and win with certainty receiving a payoff strictly greater than zero by the Generalized Cost Condition, a violation of the Expected Payoff Result. So if $P_i(k_i'; G_{-i}) = 1$ then $i = m+1$. But since $i \in \hat{N}_L$, $v_{m+1}''(k_i') > 0$ by the Strict Cost Condition. So player $m+1$ receives a strictly positive payoff, which violates the Expected Payoff Result.

Case 3: $P_i(k_i'; G_{-i}) \in (0, 1)$. By the definition of $\hat{N}_L$, the Constraint Condition and the fact that S_i is an interval, for all players $j \neq i$ with sufficiently small ε, if $[k_i' - \varepsilon, k_i') \subset S_j''$ then $[k_i', k_i' + \varepsilon] \subset S_j''$. Hence for small ε no player $j \neq i$ will put any probability on $s_j \in [k_i' - \varepsilon, k_i')$ as doing so is either infeasible or moving such probability to $s_j = k_i' + \varepsilon$, just above i's atom, will result in an increase in his probability of winning of at least $[G_i(k_i') - \lim_{z \to k_i'^-} G_i(z)] P_i(k_i'; G_{-i}) > 0$ at negligible cost, by A1. Likewise no player $j \neq i$ who has any probability of losing ties to i at k_i' will place any probability at $s_j = k_i'$ since increasing his score to $s_j = k_i' + \varepsilon$ will eliminate the non-zero probability of such ties. Hence player i's probability of victory will not decrease if he drops his atom from $s_i = k_i'$ to $s_i = k_i' - \varepsilon$. Since $P_i(k_i'; G_{-i}) > 0$ this would increase his expected payoff by the Strict Cost Condition and A1, a contradiction.

Therefore there exists at least one equilibrium of $\tilde{C}''$ in which no player places an atom at his constraint. Take such an equilibrium G and a player $i \in N_k$. Since $s_i = k_i'$ is a zero probability event in G_i, removing k_i' from S_i'' does not change the expected payoffs in G for any player at any feasible score and G_i is still a valid distribution function. Hence if we remove k_i' from S_i'' for all $i \in N_k$, G still forms an equilibrium of the resulting game which is $\tilde{C}'$.

To complete the proof we need to show that this equilibrium of $\tilde{C}'$ − a game with a special tie-breaking rule − is also an equilibrium of C' − a game with the original tie-breaking rule. This uses the same steps as Siegel (the last two paragraphs of the proof of Corollary 1). Therefore we omit that portion of the proof here to save space, and just point out that in Siegel $\tilde{C}$ is our $\tilde{C}'$, $\tilde{u}_i$ denotes player i's expected payoff in the equilibrium G of $\tilde{C}'$, and the Modified Tie Lemma applies rather than the Tie Lemma that is used in Siegel.

Q.E.D.

5. Application

Derivation of players' equilibrium expected payoffs only requires simple calculations of the reaches and powers of the players in N_w. To illustrate the use of the results consider the following application from the literature.

Meirowitz (2008) analyzes the sources of incumbency advantage in a first-past-the-post electoral contest where politicians compete in campaign spending. One dollar of campaign spending raises the score of the political candidate by one. The incumbent (I) and the challenger (C) have a common valuation of the prize which is normalized to 1. The candidates have potentially different marginal utility cost of raising funds, β_i $\forall$ $i \in \{I, C\}$. Meirowitz argues that incumbents tend to be more efficient at fundraising. As a sitting officeholder an incumbent is in a position to dispense political favors and hence has better access to resources, in which case $\beta_I < \beta_C$. Meirowitz's framework allows for a positive headstart advantage $\alpha > 0$ for the incumbent due to existing name recognition. In the analysis for spending limits with a positive headstart, Meirowitz only presents the case where the spending limit, $\overline{m}$, is so restrictive that the incumbent would win the contest even if the challenger where to spend the maximum permissible amount and the incumbent were to spend zero, $\overline{m} < \alpha$. Hence the equilibrium is in pure strategies where no candidate engages in campaign spending.

In application (*i*) below we extend Meirowitz's analysis to less restrictive spending limits where the limit does not completely curb competition, $\overline{m} > \alpha$. Pastine and Pastine (2012b) addressed this via full derivation of the players' equilibrium strategies. This example demonstrates how much simpler the task becomes using the Expected Payoff Result. In application (*ii*) we extend the analysis to elections with more than two candidates and we allow the effectiveness of spending to vary across candidates.

(*i*) A spending limit that does not completely curb competition: The main argument in favour of spending limits is that they restrict incumbents' ability to exploit their fundraising advantage - see the elegant argument from Justice Stevens in the US Supreme court case McConnell v. FEC (2003). Opponents of limits suggest that a spending limit restricts the challenger's ability to catch up with the incumbent who often enjoys a headstart advantage due to the incumbent's initial name recognition. In his dissenting opinion in McConnell v. FEC (2003), Supreme Court Justice Scalia writes: "… any restriction upon a type of campaign speech that is equally available to challengers and incumbents tends to favour incumbents." Opponents of spending limits also follow the line of logic in Stigler (1971) and suggest that incumbents would not legislate limits if the legislation did not serve them.

The Expected Payoff Result can be applied to show that with any headstart advantage, $\alpha > 0$, however small, in any equilibrium a spending limit benefits the incumbent no matter how dramatic the difference in fundraising abilities may be. The "headstart advantage" argument of the opponents of spending limits always trumps the "fundraising efficiency" argument of the proponents of limits.

In order to use the Equilibrium Existence Result we make one modification to the Meirowitz (2008) framework. In Meirowitz (2008) the contest with spending limits is weakly generic, but not strictly-generic because spending is less than or equal to the limit, a violation of the Strategy Set Condition. We require that spending must be strictly less than the limit, creating a strictly-generic contest and hence the Equilibrium Existence Result applies. In this context little is lost by the change, as continuous spending games such as this are intended as analytically tractable approximations to the discrete choice reality, where monetary units are not infinitely divisible. Since equilibrium always exists in discrete-choice games, choosing a continuous-choice approximation in which equilibrium also exists seems reasonable.

Next convert Meirowitz' framework into the notation of this paper. The monetary limit on campaign spending, $\overline{m}$, is common to both players. However, since the incumbent has a headstart advantage of $a_I = \alpha$ while $a_C = 0$, the constraints on *scores* are asymmetric: $k_I = \alpha + \overline{m}$ and $k_C = \overline{m}$. The challenger's payoff and cost functions are given by $v_C(s_C) = 1 - \beta_C s_C$ and $c_C(s_C) = \beta_C s_C$ for $s_C \in [0, k_C)$. Since the incumbent starts with a score of α his payoff function is $v_I(s_I) = 1 - \beta_I(s_I - \alpha)$ and $c_I(s_I) = \beta_I(s_I - \alpha)$ for $s_I \in [\alpha, k_I)$. Therefore the reach of the challenger is $r_C = \min\{\overline{m}, (1/\beta_C)\}$ and the reach of the incumbent is $r_I = \min\{\alpha + \overline{m}, \alpha + (1/\beta_I)\}$.

Without a spending limit the challenger is the marginal player; the reach of the challenger is lower than the reach of the incumbent, $1/\beta_C < \alpha + (1/\beta_I)$. From the Expected Payoff Result the challenger has zero expected payoff. The threshold is $1/\beta_C$, so the incumbent has an expected payoff equal to his power, $1 - \beta_I[(1/\beta_C) - \alpha] > 0$.

If the spending limit is less than $1/\beta_C$, then it is binding and $r_C = \overline{m} < (1/\beta_C)$. Since r_C is less than the incumbent's reach $r_I = \min\{\alpha + \overline{m}, \alpha + (1/\beta_I)\}$ the challenger is still the marginal player and his expected payoff remains zero. However the limit reduces the challenger's reach (the threshold of the game) and hence increases the expected payoff of the incumbent to $1 - \beta_I(\overline{m} - \alpha) > 0$. The imposition of a spending limit always benefits the incumbent as long as the incumbent has a headstart advantage however small that may be.

In addition, suppose that prior to the above game the two parties had the opportunity to increase their initial score a_i through voter registration drives. Increases in a_I would have a positive benefit for the incumbent but marginal increases in a_C would not change the challenger's expected payoff.

(*ii*) A spending limits with multiple candidates and asymmetric campaign spending effectiveness: In countries such as France and the U.K. where campaign spending limits are in place, often more than two political parties compete. Elections with more than two candidates are significantly more difficult to analyze if full derivation of the equilibrium is required. Therefore the literature largely focuses on two-candidate races as in Meirowitz (2008) and Pastine and Pastine (2012b). However since the Expected Payoff Result and the Existence of Equilibrium result do not require the full derivation of equilibrium, we can easily add more candidates and compute which political candidate benefits from a spending limit in any equilibrium. Below we employ our results in a model with multiple candidates who may have asymmetric campaign spending effectiveness. Application (*i*) already demonstrates that the "headstart advantage" argument against spending limits always dominates the "fundraising efficiency" argument in favor of spending limits if candidates have equal spending efficiency. Here we show that limits may benefit an opponent if his spending is more effective, which is often found empirically. A moderate cap on spending may benefit a charismatic third-party candidate, but a very restrictive cap benefits the incumbent.

Add a third-party candidate to the model described in application (*i*) with the same notation. Suppose that the third-party candidate (candidate L) is charismatic and has leadership skills so that one dollar of campaign spending increases his score by $\eta_L > 1$. So the third-party candidate's cost of achieving the score s_L is $\left(\frac{\beta_L}{\eta_L}\right) s_L$ and his reach is $r_L = \min\{\eta_L \overline{m}, (\eta_L / \beta_L)\}$. In order to restrict attention to the most interesting cases, assume that as a third party candidate he lacks a large fundraising base so fundraising is more onerous for him than for candidate C, $\beta_L \in \left(\eta_L \beta_C, \frac{\eta_L - 1}{\alpha}\right)$. This implicitly assumes that the range exists, *i.e.* the incumbent's headstart advantage is not too large, $\alpha < \frac{\eta_L - 1}{\eta_L \beta_C}$.

The contest is strictly generic except when parameter values are such that there are two players with reach at the threshold, which is a violation of the Generalized Power Condition. L and C have the same reach if $\overline{m} = \eta_L / \beta_L$. And L and I have the same reach if $\overline{m} = \frac{\alpha}{\eta_L - 1}$ or $\overline{m} = \left(\frac{\eta_L}{\beta_L}\right) - \alpha$. Notice that for any given $\overline{m}$, if η_L was drawn from any continuous distribution, the contest would be strictly generic with probability one. So if the spending limit is legislated before candidates' abilities are randomly drawn and become common knowledge, the existence and payoff results will apply.

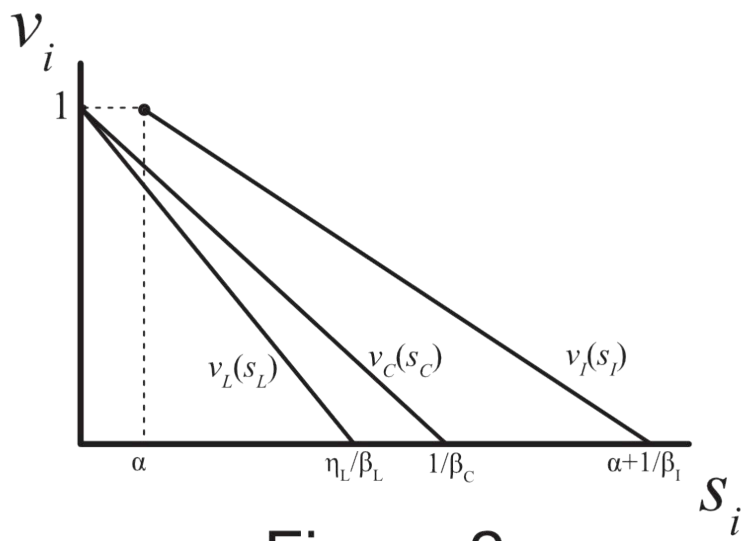

Figure 2

In the absence of a spending limit $r_I > r_C > r_L$. Since the contest is strictly generic we know that at least one equilibrium exists by the Equilibrium Existence Result. The challenger is the marginal player. He is disadvantaged because the incumbent has a head-start advantage and is a better fundraiser. By the Expected Payoff Result, in any equilibrium the incumbent has a positive expected payoff of $1 - \beta_I\left[\left(\frac{1}{\beta_C}\right) - \alpha\right] > 0$, while the challenger and the third party candidate receive an expected payoff of zero. The third-party candidate has greater effectiveness of campaign spending than either of his rivals but this is not enough to outweigh the incumbency advantage or the superior fundraising abilities of his rivals.

However, with a common monetary cap $\overline{m} < \left(\frac{1}{\beta_L}\right)$, all candidates are restricted at their score constraints and the reaches of the candidates are given by $r_I = \alpha + \overline{m}$, $r_C = \overline{m}$ and $r_L = \eta_L \overline{m}$. If the cap is moderate $\overline{m} \in \left(\frac{\alpha}{\eta_L - 1}, \frac{1}{\beta_L}\right)$, then $r_L > r_I > r_C$ as shown in Figure 3.

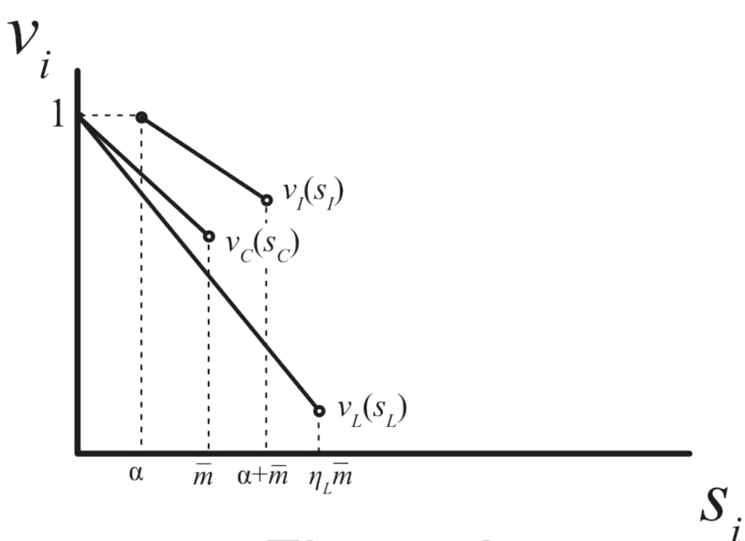

Figure 3

Although all three candidates face the same legal constraint on campaign spending, their asymmetries result in different constraints on their scores. The incumbent's head-start advantage means that his reach is higher than the challenger's. And the third-party candidate's effectiveness in spending means that his reach is higher than the challengers, and with these parameters higher than the incumbent's as well. The incumbent is now the marginal player. The threshold of the contest is $T = \alpha + \overline{m}$. By the Expected Payoff Result, the incumbent and the challenger have expected payoffs of zero and the third-party candidate receives an expected payoff of $1 - \beta_L\left(\frac{\alpha+\overline{m}}{\eta_L}\right) > 0$. With a moderate limit, the campaign spending effectiveness of the third-party candidate overwhelms the head-start advantage of the incumbent. Hence a moderate limit hurts the incumbent compared to no restrictions.

If the cap is very restrictive, $\overline{m} \in \left[0, \frac{\alpha}{\eta_L - 1}\right)$, then the order of candidates' reaches is $r_I > r_L > r_C$. The third-party candidate is the marginal player and $\eta_L \overline{m}$ is the threshold. By the Expected Payoff Result, the incumbent has the expected payoff $1 - (\eta_L \overline{m} - \alpha)\, \beta_I > 0$. The challenger and the third-party candidate have expected payoff of zero. The head-start advantage of the incumbent overwhelms the campaign spending effectiveness of the third-party candidate with leadership skills. The cap is too restrictive for the third-party candidate to catch up with the incumbent's head start. Note that the expected payoff of the incumbent in this case is higher than the expected payoff he would have had if there were no campaign spending restrictions.

6. Participation

A player is said to *participate* in an equilibrium of a contest if he chooses scores with a positive cost of losing with strictly positive probability. Here we present a very simple generalization of the results on participation for unconstrained contests to contests with constraints.

Participation Result: *In a strictly generic contest with or without constraints, if*

(i) $\quad \frac{c_{m+1}(\max\{a_{m+1}, x\})}{v_{m+1}(a_{m+1})} < \frac{c_i(x)}{v_i(a_i)} \quad$ *for all* $x \in \{S_i : x \le r_i$ and $c_i(x) > 0\}$

and

(ii) $\quad \frac{v_{m+1}(\max\{a_{m+1}, x\})}{v_{m+1}(a_{m+1})} \ge \frac{v_i(x)}{v_i(a_i)} \quad$ *for all* $x \in \{S_i : x \le r_i\}$

then player i does not participate in any equilibrium. In particular, if these conditions hold for all players in $N_L \setminus \{m+1\}$ *then only the m+1 players in* $N_w \cup \{m+1\}$ *may participate.*

Since with constraints it is possible for $\sup S_i > k_{m+1} = \sup S_{m+1}$ even when $i > m+1$, the conditions restrict attention to $x \in \{S_i : x \le r_i\}$ which implies $x < k_{m+1}$ by the Generalized Power Condition. Since in equilibrium player *i* will not exceed his reach this change is innocuous. The proof is otherwise identical to the proof of the corresponding result in Siegel and so is omitted.

So a player will not participate if for every possible score he might choose: (i) his cost of losing at that score relative to his value of winning with no effort is strictly higher than the same ratio for the marginal player and (ii) his value from winning at that score relative to his value of winning with no effort is weakly less than the same ratio for the marginal player. This simply says, unsurprisingly, that if at every possible score a player has higher (normalized) costs and a lower (normalized) valuation than the marginal player, that player will not participate.

This straightforward result may be useful for two reasons. The corresponding result for contests without constraints has been crucial in developing general algorithms for finding players' equilibrium strategies in unconstrained contests, see Siegel (2010 and 2014), and so the extension to constrained contests may also prove useful. Moreover, extensive work has been done exploring participation in contest models, see Hilman and Riley (1989), Ellingsen (1991), Baye *et al.* (1993 and 1996), Siegel (2012) and Klose and Kovenock (2015). One of the major issues in this literature is finding conditions under which only *m+1* players participate in an (unconstrained) all-pay auction. The Participation Result provides sufficient conditions for these results to generalize to contests with constraints. This can be seen by considering a well-known example from the literature.

Example 4: Take the following model from Hillman and Riley (1989) and Baye *et al.* (1993 and 1996). There is $m=1$ prize and $n\geq 3$ players with $v_i(s_i) = V_i - s_i$, $c_i(s_i) = s_i$ and $S_i = [0, \infty)$. Ties are decided with even probability. Denote the players A, B, C *etc.* and assume $V_A > V_B > V_C > V_D \geq \cdots \geq V_n$.[3] Hillman and Riley (1989) shows that there exists an equilibrium where only the two players with the highest valuation participate and Baye *et al.* (1993 and 1996) shows that there does not exist any equilibrium where any other player or players participate. These results can also be seen by straightforward application of the Equilibrium Existence Result and the Participation Result since player B is the marginal player and for $i \in \{C, D, \ldots n\}$ (*i*) $\frac{x}{V_B} < \frac{x}{V_i}$ and (*ii*) $\frac{V_B - x}{V_B} \geq \frac{V_i - x}{V_i}$ for all $x \in (0, V_i]$.

Now impose a constraint on player B so that $S_B = [0, k_B)$. Consider first a constraint in the range $k_B \in (V_C, \infty)$. The introduction of the constraint is not innocuous when $k_B \in (V_C, V_B)$; The constraint changes both the equilibrium and the expected payoff for player A. This can be seen from the Expected Payoff Result which, since the threshold is k_B, yields an expected payoff to player A of $V_A - k_B > 0$ which implies that the equilibrium strategies of the players are altered by the existence of the constraint. Nevertheless, the participation results of Hillman and Riley (1989) and Baye *et al.* (1993 and 1996) are robust to the imposition of a constraint on player B in the range $k_B \in (V_C, \infty)$. For $k_B \neq V_C$ the game is strictly generic and hence from the Equilibrium Existence Result we know that at least one equilibrium exists. For $k_B > V_C$ player B is still the marginal player, and hence direct application of the Participation Result implies that only the m+1 players in $N_w \bigcup \{m+1\}$ participate, so only A and B participate in any equilibrium.

However, in the range $k_B \in (0, V_C)$ more than $m+1$ players may participate. When the constraint is in this range, player C is the marginal player and the threshold is V_C. Application of the Participation Result implies that players $D, E \ldots n$ do not participate in any equilibrium, so they all choose scores of zero with probability one. However participation by player B cannot be ruled out as $\frac{c_{m+1}(\max\{a_{m+1}, x\})}{v_{m+1}(a_{m+1})} = \frac{x}{V_C} > \frac{c_B(x)}{v_B(a_i)} = \frac{x}{V_B}$ for all $x \in (0, k_B)$.

In fact for $k_B \in (0, V_C)$ all three must participate. Player A$\in N_w$ will participate since the Generalized Threshold Lemma shows he has a score in the support of his strategy in G that approaches or exceeds the threshold. Since A's payoffs from both winning and losing are strictly decreasing in his score, the only reason he would choose such a high score is that C has a score in the support of his strategy in G that approaches the threshold. So C participates as well. Now conjecture that there exists an equilibrium where player B does not participate. In this equilibrium player B chooses $s_B = 0$ with certainty and his probability of winning is arbitrarily close to zero by the Modified Zero Lemma. As shown in Hillman and Riley (1989) and Baye *et al.* (1993 and 1996) the two-player game with just players A and C has unique equilibrium distribution functions of $G_A(x) = x / V_C$ and $G_C(x) = (V_A - V_C + x) / V_A$ for $x \in [0, V_C]$. With these strategies, a score of zero has zero probability of winning. Hence in the three-player game if player B plays his conjectured pure strategy of $s_B = 0$, players A and C will play according to the Hillman and Riley strategies. However, given these conjectured equilibrium strategies, in the full game with many players, if player B chooses $s_B \in (0, k_B)$ he gets an expected payoff of $G_A(s_B)\, G_C(s_B)\, V_B - s_B = \frac{s_B}{V_A V_C}\left[\, V_B(V_A - V_C) - s_B(V_B - V_A V_C)\right]$ which is greater than zero for sufficiently small s_B. So player B has a profitable deviation which contradicts the Expected Payoff Result. Hence, when $k_B \in (0, V_C)$ in any equilibrium all three players A, B and C must participate.

7. Conclusion

In this paper, we analyze constraints on players' choices in a broad class of all-pay auctions which incorporates contests with many players and multiple prizes, contests with conditional investments, head starts and non-ordered payoff functions. In the first main contribution of the paper we derive simple closed-form formulae for players' expected payoffs in any equilibrium where some, all or none of the players are constrained. The formulae are straightforward to calculate and do not require the derivation of the equilibrium or equilibria.

In the second main contribution of the paper we employ the Expected Payoff Result to prove the existence of equilibrium. This is not-trivial since player payoffs are discontinuous in their pure-strategies and there are infinitely many pure-strategies.

Together these results mean that in applications one can easily calculate player expected payoffs in all-pay contests with constraints, bypassing the need for a full characterization of the equilibrium or equilibria. In some applications, the expected value of the contest to the players may be the main item of interest. For instance the question may concern the impact of a policy change on the players in equilibrium, such as the relaxation of a liquidity constraint, imposition of a binding deadline, a salary cap, or utilization of an affirmative action policy. The expected value of the contest to the players is also potentially useful in analyzing players' incentives to invest in relaxing their constraints prior to the contest. We show that no player has any incentive to marginally relax his constraint. A relaxation of a player's constraint is only beneficial to him if it is a significant enough change to allow him to have a higher reach than all but $m-1$ of his competitors and further relaxation has no benefit to him.

In other applications where full characterization of the equilibrium is of interest, calculation of players' expected values from the contest is the first crucial step since typically all-pay contests have equilibrium in mixed strategies.

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Notes

[1] See, among others, Hillman and Samet (1987), Hillman and Riley (1989), Ellingsen (1991), Baye *et al.* (1993), and Konrad (2002) for contests in rent seeking; Che and Gale (2006), Kaplan and Wettstein (2006), Pastine and Pastine (2013) and Szech (2015) for political contests; Bond (2009) for litigation contests; Clark and Riis (1998) for job tournaments; Pastine and Pastine (2011) for advertising competition; Fu (2006) and Pastine and Pastine (2012a) for affirmative action in college admissions. See also Che and Gale (1996), Laffont and Robert (1996), Gavious *et al.* (2002), Dekel *et al.* (2007), Sahuguet (2006), Kirkegaard (2008), and Pai and Vohra (2014) for frameworks with incomplete information and constrained players. See Rapoport and Amaldoss (2000) for an experimental analysis of all-pay auctions with bid caps and the comment in Dechenaux *et al.* (2006). See Megidish and Sela (2014) for constraints in a sequential contest. See Konrad (2009) for an extensive survey on contests.

[2] In order to emphasize the dependence of player *i*'s probability of winning and expected utility on the strategies of the other players we've altered the notation in Siegel slightly here. When following the proofs in Siegel it is useful to note that our $P_i(s_i; G_{-i})$ is the same as Siegel's $P_i(s_i)$ and our $u_i(s_i; G_{-i})$ is the same as Siegel's $u_i(s_i)$.

[3] The authors also permit equality of valuation for the first four players. Here we restrict the domain in order to ensure a strictly generic game throughout the example.