MEASURING INTERGENERATIONAL MOBILITY AND EQUALITY OF OPPORTUNITY

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ABSTRACT

This paper explores the link between the measurement of intergenerational mobility and the notion of equality of opportunity. We show how recently proposed theories of equality of opportunity can be meaningfully adapted to the intergenerational context. This throws a new light on the interpretation of existing mobility measures: these may be interesting to measure mobility as movement, but they are inadequate to capture the notion of equality of opportunity. We propose some new mobility measures, which start from the idea that the intergenerational transition matrix gives useful information about the opportunity sets of the children of different social classes. These measures are used in an empirical illustration to evaluate the degree of inequality of opportunity in the US, Great Britain and Italy.

This paper benefited from comments of the participants at a meeting of the HCM-Network "Distribution and Redistribution of Income" (Bordeaux) and at a conference on "Egalitarianism and Responsibility" (Cergy). Comments by Marc Fleurbaey, Tony Shorrocks and Alain Trannoy were especially helpful.

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1 Introduction

One of the most obvious applications of the idea of "equality of opportunity" is in an intergenerational context. The feeling is widespread that income inequalities which arise from differences at birth are a sign of unequal opportunity and should definitely be a cause of ethical concern. There is less consensus about the desirability of further government policy to reduce inequality in outcomes, due to other factors[1]. Such a concern for equality of opportunity is apparently one of the main reasons to be interested in intergenerational mobility: our immediate intuitions suggest that more intergenerational mobility means more equal opportunities. There may be other reasons to be interested in mobility (see, e.g., Atkinson, 1981). Mobility may be an objective in its own right. Or it may be instrumental in leading to greater efficiency. Or (Atkinson's own proposal) it may influence the overall level of social welfare, defined over the distribution of income for different generations. But, although all these other reasons may play a role, there will almost always be a link with concern for equality of opportunity among children from different income classes.

Since intergenerational mobility is an old concern in both sociology and economics, various statistical and normative measures have been proposed in the literature. The axiomatic properties of these measures have been investigated by Shorrocks (1978, 1993). He shows that there are fundamental incompatibilities between at first sight quite reasonable axioms. These axioms can be related to the different reasons for being interested in intergenerational mobility mentioned before.

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1 Two recent revealing quotes -among many which could be given- are the following: "I am going to take the position that if economic success is largely unpredictable on the basis of observed aspects of family background, than we can reasonably claim that society provides equal opportunity. There might still be significant inequality in income across individuals, due to differences in ability, hard work, luck, and so on, but I will call these unequal outcomes. On the other hand, if economic success is highly predictable on the basis of family background, then I think it is difficult to accept the claim that our society provides equal opportunity" (Stokey, 1996, 2). And: "If the industrious and talented have much higher incomes than the work-shy and stupid, then not only might we not be worried about inequality between them, but also such inequality as there is might be considered a positive good. If, on the other hand, inequality arises from chance of birth -if one's income is virtually determined by that of one's parents- then an unequal distribution might be a cause of serious concern" (Johnson and Reed, 1996). Both citations are revealing in that they do not rank "ability" and "talent" among the factors to be compensated for. This is of course debatable.

With this paper we want to explore the link between the measurement of equality of opportunity and intergenerational mobility. In section 2 we discuss how recently developed theories of equality of opportunity (Bossert, 1995; Bossert et al., 1996; Fleurbaey, 1995a, 1995b; Roemer, 1993, 1996) can be interpreted in an intergenerational context and we point to some basic normative choices which have to be made. Either one concentrates on the overall evaluation of the opportunities of children from different descent or one tries to realise equal outcomes for all children who exert the same effort. Both intuitions are basically incompatible. In section 3 we discuss the existing mobility measures based on transition matrices and we will point out the relationship between different axioms and different motivations to be interested in mobility. It will turn out that none of the existing measures captures adequately the basic intuitions of equality of opportunity. In section 4 we show how the alternative measures proposed in section 2 can be interpreted and applied for the analysis of transition matrices. We also present an empirical illustration. Section 5 concludes.

2 Equality of opportunities in an intergenerational context

It is common in the analysis of intergenerational mobility to concentrate on the two-period (or two-generations) case and to represent the economic status of all individuals by a scalar measure, i.e. to neglect all aspects of life-time mobility. The basic material for the empirical analysis then consists of |N| parent-child pairs, with the respective income levels: $(\tilde{y}_i^1, \tilde{y}_i^2) \in \Re^{2+}, i = 1, \ldots, |N|$. Starting from the overall vector of incomes of two generations $\tilde{y} = (\tilde{y}_1^1, \tilde{y}_1^2, \ldots, \tilde{y}_i^1, \tilde{y}_i^2, \ldots, \tilde{y}_{|N|}^1, \tilde{y}_{|N|}^2)$, one then divides the income vectors for parents and children in n equally-sized quantiles and one constructs the (*nxn*) bistochastic matrix P, where p_{ij} is the proportion of children with a parent income in quantile i who themselves have an income in quantile j (of the distribution of child incomes). This matrix P is called a quantile transition matrix[2]. We will return to the evaluation of such matrices in the next section. In this section we will concentrate on the evaluation of the basic income vector $\tilde{y}$ from the point of view of equality of opportunity.

The crucial feature of the various theories of "equal opportunities" is the way in which they distinguish "opportunities" and "outcomes". Recent theories (the most prominent economic examples being Roemer, 1996, Bossert, 1995, and Fleurbaey, 1995a, 1995b) draw a distinction between two sets of variables: on the one hand variables for which the individual (in our context the child) cannot be held responsible, on the other hand variables for which she is responsible. The basic idea is to compensate for differences in outcomes resulting from the former variables, but to leave intact outcome differences resulting from the latter variables. Roemer (1993, 1996) describes a specific procedure to implement this distinction. He proposes to partition the population in groups which are homogeneous w.r.t. the non-responsibility characteristics. All individuals in the same group are said to be of the same "type". By definition, within each type the differences in outcomes can then be ascribed to differences in factors for which the individuals are responsible. Very often these responsibility variables are basically unobservable (e.g. effort level). Roemer formulates a concrete proposal to overcome this difficulty. He assumes that (within types) the individual outcome is a monotonically increasing function of an (unobservable) "effort"-variable z. Two people of different types are said to have exercised the same degree of responsibility if they are at the same percentile of the distribution of outcomes for their type.

It is straightforward to translate these ideas to the intergenerational setting. We concentrate on the situation of the children and we assume that their "type" is defined by the income of their parent. If we suppose that there are n levels of parental income, we will have n different types. Suppose that there are m children associated with each level of parental income, i.e. $nm=|N|$.

2 In theoretical work, and more specifically in the analysis of Markov-chains, the basic concept is an analogous transition matrix but instead of quantiles one works with a (fixed) vector of n income levels $[\bar{y}_1, \ldots, \bar{y}_n]$ (with $\bar{y}_1 < \bar{y}_2 < \ldots < \bar{y}_n$). An element p_{ij} in P then gives the probability that a parent with an income $\bar{y}_i$ has a child with an income $\bar{y}_j$.

In the terminology of the theories of equality of opportunity, we now have n types and m degrees of responsibility. It is convenient for our purposes to summarize the information which this structure yields by the (nxm)-matrix Y:

$$Y = \begin{bmatrix} y_{1(1)} & \dots & y_{1(\rho)} & \dots & y_{1(m)} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ y_{t(1)} & \dots & y_{t(\rho)} & \dots & y_{t(m)} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ y_{n(1)} & \dots & y_{n(\rho)} & \dots & y_{n(m)} \end{bmatrix}$$

where the incomes in each row are ranked in increasing order[3]. We will interpret the differences within each row as resulting from differences in effort. This implies that differences in outcomes following from differences in natural abilities (not captured by the differences in parent's incomes) are treated here as within the children's responsibility. As noted before, this is extremely debatable from a broader philosophical point of view. Yet, although the one-dimensional case can easily be generalised to a setting where the types are defined on the basis of parent's income plus other variables, this broader framework would require us to go beyond the information which is traditionally summarized in transition matrices. The first purpose of this paper is the evaluation of such matrices. Note indeed the close relationship between Y and the transition matrices P: if we define n quantiles of children's incomes, the latter (nxn)-matrix can be immediately derived from the former (nxm)-matrix. This analogy will be exploited later on.

For simplicity, we will assume that the ordering of matrices Y on the basis of the degree of equality of opportunities can be represented by a function $S[Y]$. It seems reasonable to impose some monotonicity axiom on this function. We present it in a strong and a weak form:

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3 It is easy to adapt the framework to the case of a different number of children in each row: we then simply follow Roemers's percentile approach.

WRI (Weak Responsiveness)

Let Y and $\tilde{Y}$ be such that $\tilde{y}_{i(j)} = y_{i(j)} \quad \forall i(j) \neq k(l) \quad \tilde{y}_{k(l)} = y_{k(l)} + \varepsilon \quad (\varepsilon > 0)$. Then $S[\tilde{Y}] \geq S[Y]$.

SRI (Strong Responsiveness)

Let Y and $\tilde{Y}$ be such that $\tilde{y}_{i(j)} = y_{i(j)} \quad \forall i(j) \neq k(l) \quad \tilde{y}_{k(l)} = y_{k(l)} + \varepsilon \quad (\varepsilon > 0)$. Then $S[\tilde{Y}] > S[Y]$.

Let us now turn to the interpretation of the equality of opportunity-concept. For each type *t*, *t=1,...,n,* we can say that the corresponding row of the matrix *Y*, i.e. the vector $y_{t(.)} = [y_{t(1)} \ldots y_{t(\rho)} \ldots y_{t(m)}]$ describes the *opportunities of type t*. For each degree of responsibility ρ, ρ*=1,...,m,* we can consider the column $y_{.(\rho)} = [y_{1(\rho)} \ldots y_{t(\rho)} \ldots y_{n(\rho)}]$. This vector describes the *outcomes for different types at the same level of responsibility*. All this suggests that there are two alternative ways to structure the evaluation function $S[Y]$, summarized in the following two axioms:

SER (Separability over Responsibility):

$$S[Y] = F[u_1[y_{.(1)}], u_2[y_{.(2)}], \ldots, u_m[y_{.(m)}]]$$

SET (Separability over Types):

$$S[Y] = G[v_1[y_{1(.)}], v_2[y_{2(.)}], \ldots, v_n[y_{n(.)}]]$$

These separability assumptions can be related to two basic intuitions concerning equality of opportunity. The first is made explicit in the proposal of Roemer (1996). Look at all the elements in one column of *Y*: these give the income levels reached by children of different types (different parent's incomes) but who exert the same effort level. From the point of view of equal opportunities, it seems natural to prefer a situation in which those who exercised the same effort level receive the same outcome and this completely independent of the income of their parents. Given SER, this basic idea is translated in the following axiom which says that a more equal distribution of outcomes for children at the same effort level is to be preferred[4]:

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4 We will formulate the basic insights of equality of opportunity in terms of *strong* axioms. The weak analogues can immediately be formulated and lead to obvious changes in the following theorems.

IAWR (Inequality Aversion within Degree of Responsibility)

Assume SER. Let Y and $\tilde{Y}$ be such that

$$u_i[\tilde{y}_{.(i)}] = u_i[y_{.(i)}] \quad \forall i \neq \rho, \tilde{y}_{j(\rho)} = y_{j(\rho)} \quad \forall j \neq k, l$$

$$\tilde{y}_{k(\rho)} = y_{k(\rho)} + \varepsilon, \tilde{y}_{l(\rho)} = y_{l(\rho)} - \varepsilon(\varepsilon > 0), \tilde{y}_{k(\rho)} \leq \tilde{y}_{l(\rho)}, \tilde{y}_{k(\rho+1)} \geq \tilde{y}_{k(\rho)}, \tilde{y}_{l(\rho)} \geq \tilde{y}_{l(\rho-1)}$$

$$\Rightarrow S[\tilde{Y}] > S[Y]$$

This axiom basically is a Pigou-Dalton transfer axiom, where the domain of ε is restricted by the condition that the transfer should not change the rank order in the different rows.

There is a second intuition concerning equal opportunities, however. Take each row as a description of the opportunity set of the corresponding type: the incomes which children of a given parental income class can reach by varying their level of effort (for which they are responsible). Assuming SET, these opportunities are evaluated by the functions $v_1, \ldots, v_n$. To avoid unneccessary complications, we assume that these functions are measurable on an absolute scale and hence are fully comparable over the different types. Moreover they are strictly monotone and unbounded above and below. The idea that we prefer a more equal distribution of opportunities can then be represented by the following axiom:

IABT (Inequality Aversion Between Types)

Assume SET. Let Y and $\tilde{Y}$ be such that

$$v_t[\tilde{y}_{t(.)}] = v_t[y_{t(.)}] \quad \forall t \neq k, l,$$

$$v_k[\tilde{y}_{k(.)}] = v_k[y_{k(.)}] + \varepsilon, v_l[\tilde{y}_{l(.)}] = v_l[y_{l(.)}] - \varepsilon, \varepsilon > 0$$

$$v_k[\tilde{y}_{k(.)}] \leq v_l[\tilde{y}_{l(.)}]$$

$$\Rightarrow S[\tilde{Y}] > S[Y]$$

Axiom IABT is again a kind of transfer principle, stating that a "redistribution of opportunities" is positively valued. It entails a comparison *between the rows* of the matrix *Y*.

At first sight both these approaches are plausible and capture obvious intuitions. Yet it is easy to see that they are incompatible. According to IAWR any redistribution of income from rich to poor *within* a column must be positively evaluated, even if (when we consider the rows) the redistribution goes from a low opportunities-type to a high opportunities-type. The basic dilemma we face is illustrated in Figure 1, where the effort-level z is put on the horizontal axis and the outcome on the vertical axis. Each graph represents the opportunities of a different type. In the upper part of the figure children of type (social class) j will reach a higher income for all effort levels but the lowest ones. According to IAWR a change in both the opportunities of the types i and j in the direction of the dotted line will be an improvement: this goes strongly against the intuition of IABT, because after that change the opportunities of children of type i are (weakly) dominated by the opportunities of children of type j at all effort levels. On the other hand, in the lower part of the figure the opportunities of type j-children are worse, except for the very high effort levels and at these levels there is indeed a very unequal treatment of the different types. According to IABT a further increase in this inequality (improving the opportunities of type j-children) is positively evaluated. Of course, this is not so for IAWR.

Figure 1 about here

The consequences for the specification of $S[Y]$ are shown clearly in the following theorem, which can be proven by application of standard results:

Theorem 1. (a) $S[Y]$ **satisfies WRI, SET and IABT if and only if it can be written as** $S[Y] = G[v_1[y_{1(.)}], v_2[y_{2(.)}], \ldots, v_n[y_{n(.)}]]$**, where** $G[.]$ **is non-decreasing in its arguments and constant sum strictly quasi-concave, while** $v_k[.]$ **is non-decreasing.**

(b) $S[Y]$ **satisfies WRI, SER and IAWR if and only if it can be written as** $S[Y] = F[u_1[y_{.(1)}], u_2[y_{.(2)}], \ldots, u_m[y_{.(m)}]]$**, where** $F[.]$**is non-decreasing in its arguments, while** $u_\rho[y_{.(\rho)}]$ **is non-decreasing and constant sum strictly quasi-concave.**

We are really at a crossroads here. Either we look at the different outcomes within one column, or we concentrate on the evaluation of the different rows. As mentioned already, the first road has been taken by Roemer (1996). The second road was first described in Van de gaer (1993) and further discussed in Bossert et al. (1996)[5]. In these specifications the separability axioms SER and SET are replaced by the stronger additivity axioms ADBR and ADBT respectively:

ADBR (Additivity Between Degrees of Responsibility)

$$S[Y] = \Omega\left[\sum_{\rho=1}^{m} u_\rho[y_{.(\rho)}]\right]$$

ADBT (Additivity Between Types)

$$S[Y] = \Theta\left[\sum_{t=1}^{n} v_t[y_{t(.)}]\right]$$

Moreover, to arrive at his concrete formula, Roemer (1996) strengthens the axiom IAWR[6]:

EIAWR (Extreme Inequality Aversion Within Responsibility)

Assume SER. Then $u_\rho[y_{.(\rho)}] = \min_t w_{t(\rho)}[y_{t(\rho)}] \quad \forall\rho$, where $w_{t(\rho)}[.], t = 1, \ldots, n, \rho = 1, \ldots, m$

is a non-decreasing function.

We can then formulate the following lemma, which is proven in appendix 3.

Lemma 1 (Roemer). $S[Y]$ **satisfies WRI, ADBR and EIAWR if and only if it can be written as** $S[Y] = \Omega\left[\sum_{\rho=1}^{m} \min_t w_{t(\rho)}[y_{t(\rho)}]\right]$**, where** $\Omega[.]$ **and** $w_{t(\rho)}[.]$ **are non-decreasing.**

To arrive at his concrete functional form, Van de gaer (1993) imposes next to ADBT an additional additivity assumption within types. More interesting is the anonimity condition, to which we will return later on:

5 Bossert et al. (1996) also give an axiomatisation of the concrete measures following from both approaches. Our theorem 1 is formulated at a more abstract level.

6 The axiom EIAWR is not really a "strengthening" of IAWR in its strong version, since income transfers above the minimum will have no effect on the value of $u_\rho[.]$.

ADWT (Additivity Within Types)

Assume SET. Then $v_t[y_{t(.)}] = \psi_t\left[\sum_{\rho=1}^{m} \eta_{t(\rho)}[y_{t(\rho)}]\right]$

ANT (Anonymity w.r.t. types)

Let σ_n be a permutation of {1,...,n}. Let

$$Y = \begin{bmatrix} y_{1(.)} \\ . \\ y_{t(.)} \\ . \\ y_{n(.)} \end{bmatrix}, \tilde{Y} = \begin{bmatrix} y_{\sigma_n[1](.)} \\ . \\ y_{\sigma_n[t](.)} \\ . \\ y_{\sigma_n[n](.)} \end{bmatrix}.$$

Then $S[\tilde{Y}] = S[Y]$

All these axioms lead to

Lemma 2 (Van de gaer). $S[Y]$ **satisfies WRI, ANT, IABT, ADBT, ADWT if and only if it can be written as** $S[Y] = \Theta\left[\sum_{t=1}^{n} \psi\left[\sum_{\rho=1}^{m} \eta_\rho[y_{t(\rho)}]\right]\right]$**, where** $\Theta[.], \psi[.], \eta_\rho[.], \rho = 1, \ldots, m$ **are non-decreasing, and** $\psi[.]$ **is strictly concave.**

The proof of lemma 2 can be found in Appendix 3.

Let us return to the interpretation of ANT. This axiom seems especially plausible in the between-types approach and if we focus only on the information which is available in *Y* or in the traditional transition matrices. Why would we treat children of different descent differently if their opportunities are exactly the same? The social objective is to equalise the opportunities of all children and *not* to discriminate against the children of richer parents[7]. But there is an attractive interpretation in the within-column approach also: why would the evaluation of a given income level be different for different types if they exert exactly the same effort? It is therefore useful to note that Roemer's evaluation function in lemma 1 satisfies ANT iff $w_{t(\rho)}[.] = w_{\rho}[.] \quad \forall t$.

For the analysis of transition matrices, we will strengthen axiom IABT. This axiom has been formulated at the level of the evaluated opportunities for the different types. It states that a "redistribution of opportunities" is positively valued by an ethical observer who wants to equalise opportunities. This is not sufficient to capture the idea that a redistribution of income from the better off types to the worse off types is necessarily positively valued: the valuation of such redistribution will depend on the specific functional form chosen for the *v*-functions. The stronger intuition about the desirability of redistribution of outcomes is formalised in the following axiom IABTY. The formulation of this axiom is somewhat complicated by the fact that we have to keep the incomes in each row ranked in increasing order: remember the definition of *Y*. The child that receives the transfer is in the α -th position before the transfer takes place, and in the β -th position after the transfer, $\beta \geq \alpha$. Similarly, the child that pays for the transfer is in the γ -th position before the transfer takes place, and in the δ -th position after the transfer, $\delta \leq \gamma$.

IABTY (Inequality Aversion Between Types' Incomes)

Assume SET. Let Y and $\tilde{Y}$ be such that $v_t[\tilde{y}_{t(.)}] = v_t[y_{t(.)}] \quad \forall t \neq k, l,$

$\tilde{y}_{k(\beta)} = y_{k(\alpha)} + \varepsilon$ and $\tilde{y}_{l(\delta)} = y_{l(\gamma)} - \varepsilon$

$\tilde{y}_{k(\rho)} = y_{k(\rho)}, \forall \rho < \alpha$ and $\forall \rho > \beta$; if $\alpha < \beta$, then $\tilde{y}_{k(\rho)} = y_{k(\rho+1)}, \alpha \leq \rho < \beta$

$\tilde{y}_{l(\rho)} = y_{l(\rho)}, \forall \rho < \delta$ and $\forall \rho > \gamma$; if $\delta < \gamma$, then $\tilde{y}_{l(\rho)} = y_{l(\rho-1)}, \delta < \rho \leq \gamma$

$v_k[\tilde{y}_{k(.)}] \leq v_l[\tilde{y}_{l(.)}]$

Then $S[\tilde{Y}] > S[Y]$

7 However, the desirability of such a differential treatment lies at the heart of another approach to mobility: the dynastic one, as defended by Atkinson (1981) and Dardanoni (1993). Here an income increase for a child is valued less if her parent is richer. The motivation behind this approach is to maximise welfare over the generations and surely not to equalise opportunities.

The move from IABT to IABTY has important consequences. More specifically, if we impose no further restriction on the valuation function and we want it to satisfy (weak) IABTY together with SRI, then it has to take a particular form. We can indeed show that

Lemma 3. **Assume SET. $S[Y]$ satisfies SRI, ANT and (weak) IABTY for all increasing functions $v[.]$ if and only if it can be written as $S[Y] = \Lambda[v[y_{1(.)}], v[y_{2(.)}], \ldots, v[y_{n(.)}]]$, where $\Lambda[.]$ is the leximin ordering.**

The proof of lemma 3 is in Appendix 3. The intuitive reason for the result is that, because of SRI, the transfer described by IABTY increases $v[y_{k(.)}]$ and decreases $v[y_{l(.)}]$ but that the amounts with which these valuations change can be arbitrarily big or small if no further restrictions are imposed on the $v[y_{t(.)}]$ -function.

Things get even more complicated when we want to apply the IABTY-logic to a framework with *bistochastic* matrices of transition. In that case, we will have to restrict further the kind of transfers described in the IABTY-axiom, to make sure that the matrix resulting from the transfer is still bistochastic. More precisely, we have to make sure that the child that "receives" the transfer and the child that "pays" for the transfer simply swap income classes: after the transfer the former must end in the income class in which the latter was before the transfer and vice versa. Hence the transfer has to be equal to $y_{l(\gamma)} - y_{k(\alpha)}$. As in the formulation of IABTY, we consider again the situation where the child that receives the transfer is in the α -th position before the transfer takes place, and in the β -th position after the transfer, $\beta \geq \alpha$. Similarly, the child that pays for the transfer is in the γ -th position before the transfer takes place, and in the δ -th position after the transfer, $\delta \leq \gamma$. We can then reformulate IABTY in the context of bistochastic matrices as follows:

IABTYBM (Inequality Aversion Between Types' Incomes applied to Bistochastic Matrices)

Assume SET. Let Y and $\tilde{Y}$ be such that $v_t[\tilde{y}_{t(.)}] = v_t[y_{t(.)}] \quad \forall t \neq k, l,$

$\tilde{y}_{k(\beta)} = y_{l(\gamma)}$ and $\tilde{y}_{l(\delta)} = y_{k(\alpha)}$

$\tilde{y}_{k(\rho)} = y_{k(\rho)}, \forall \rho < \alpha$ and $\forall \rho > \beta$; if $\alpha < \beta$, then $\tilde{y}_{k(\rho)} = y_{k(\rho+1)}, \alpha \leq \rho < \beta$

$\tilde{y}_{l(\rho)} = y_{l(\rho)}, \forall \rho < \delta$ and $\forall \rho > \gamma$; if $\delta < \gamma$, then $\tilde{y}_{l(\rho)} = y_{l(\rho-1)}, \delta < \rho \leq \gamma$

$v_k[\tilde{y}_{k(.)}] \leq v_l[\tilde{y}_{l(.)}]$

Then $S[\tilde{Y}] > S[Y]$

This axiom will play an important role in the following section, in which we turn to the evaluation of transition matrices $P \in \Gamma$, where Γ is the class of bistochastic transition matrices. The link between the definitions of P and Y is obvious and has been explained before. The effects of the transfer described in IABTYBM can then be interpreted in terms of a transformation of the transition matrix. Assuming, without loss of generality, that the child with a parent from income class k had an income in the j-th class, and that the income of the child with a parent from income class l belonged itself to the h-th class, the transfer will transform the original matrix P into $\tilde{P}$, where

$$\tilde{p}_{kj} = p_{kj} - (1/m); \tilde{p}_{kh} = p_{kh} + (1/m); \tilde{p}_{lj} = p_{lj} + (1/m); \tilde{p}_{lh} = p_{lh} - (1/m) \quad \text{and } j < h.$$

3 Social mobility and the intergenerational transition matrix

In general, we define a mobility index as a function $M{:}\Gamma \rightarrow \Re{:}P \rightarrow M[P]$, where $P = [p_{ij}] \in \Gamma$. Many different mobility indices have been proposed in the sociological and economic literature. Some of them are described in Appendix 1. In Table 1 we show their values for a sample of published empirical transition matrices, which are described in more detail in Appendix 2. No unambiguous ranking is possible on the basis of these measures. Except for one measure, B(US) (for the US and taken from Behrman and Taubman, 1985) is the most mobile matrix. It can also be defended that the transition matrix A(GB), calculated for Great Britain by Atkinson et al. (1983) is probably the least mobile matrix. But it is not clear how the other matrices should be ordered.

Table 1 about here

If different measures give different rankings, it becomes important to understand better their normative implications. Shorrocks (1978) proposed the following axioms which reasonable mobility indices are supposed to meet[8]:

I (Immobility)

$$\forall P \in \Gamma: M[P] \geq M[I].$$

PM (Perfect Mobility)

$$Let \quad P^M = \tfrac{1}{n}\iota\iota'. \quad Then \quad \forall P \in \Gamma \neq P^M: \quad M[P^M] > M[P]$$

SM (Shorrocks Monotonicity)

If $\tilde{p}_{ij} \geq p_{ij} \forall i \neq j$ and $\exists \tilde{p}_{ij} > p_{ij}$, then $M[\tilde{P}] > M[P]$.

Since we will later concentrate on the evaluation of transformations of the matrix P, it is useful to reformulate SM as:

If $\tilde{P}$ can be obtained from P through a finite sequence of MADT, then $M[\tilde{P}] > M[P]$, where ***MADT*** is a "movement away from the diagonal transformation", defined as

$$\tilde{p}_{ij} = p_{ij} + \varepsilon; \tilde{p}_{ji} = p_{ji} + \varepsilon; \tilde{p}_{ii} = p_{ii} - \varepsilon; \tilde{p}_{jj} = p_{jj} - \varepsilon \quad (\varepsilon > 0)$$

The first axiom states that the completely immobile transition matrix is the unity matrix. This axiom is widely accepted and identifies a lower bound for mobility. The next axiom provides an upper bound. Mobility is maximal if children from all classes have an equal probability to achieve each possible outcome. The third axiom captures the following intuition: if the elements on the diagonal of $\tilde{P}$ are smaller than those on the diagonal of P, then there is more movement between income classes in $\tilde{P}$ than in P, and therefore $M[\tilde{P}]$ should be larger than $M[P]$. Clearly, SM implies I.

8 I is the identity matrix, ι is a vector with all elements equal to 1.

All these axioms look very plausible. However, Shorrocks (1978) has drawn attention to the basic conflict between SM and PM. For later reference we will summarize this result as

***Shorrocks (1978) impossibility theorem.* SM and PM are incompatible.**

The basic conflict between these two axioms urges us to reflect more about the basic reasons to be interested in intergenerational mobility and on the relationship between these basic reasons and the exact formulation of the axioms. Shorrocks (1978, p. 1016) interprets his result as reflecting a conflict between mobility as movement (captured by SM) and mobility as lack of predictability (captured by PM). We see a more basic conflict between mobility as movement and mobility as a means to equalize opportunities. The axiom SM fits perfectly into the former interpretation, but PM does not. It is not straightforward at all that equal rows of the transition matrix would reflect maximal movement. Compare P^M and B in the following two-class example:

$$P^M = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

It can be argued that there is much more movement with the matrix B. On the other hand, PM is exactly in line with the equal opportunity ideas in the previous section in both interpretations (as described in Lemma 1 and Lemma 2): P^M is the best possible matrix because it incorporates a completely equal distribution of opportunities. But what then about SM? What is wrong with the basic intuition that more movement leads to more equal opportunities? To get a better insight into this problem, we concentrate on the axioms ANT and IABTYBM.

<u>Anonymity Between Types</u>

ANT can easily be reformulated for the analysis of transition matrices:

ANT* (Anonymity Between Types)

Let $\mathrm{E}[P]$ be any matrix obtained from P by permuting rows of P. Then $M[\mathrm{E}[P]] = M[P]$.

It is easy to show

Theorem 2. **SM and ANT* are incompatible.**

Proof: Consider the mobility matrix

$$P_\varepsilon = \begin{bmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{bmatrix}$$

The value of the mobility index for P_ε can now be written as $M[\varepsilon]$. By SM, $M[2/3] > M[1/3]$. By ANT*, $M[2/3] = M[1/3]$.

The conflict between ANT* and mobility as movement (captured by SM) can further be illustrated by the immediate result that ANT* and I imply PP (perfect predictability), formalised as follows

PP (Perfect Predictability)

Let $\mathrm{E}[I]$ be any matrix obtained out of I by permuting rows. Then $M[\mathrm{E}[I]] = M[I]$.

This implies that the matrix *B*, introduced before, has "minimal mobility", because the positions of all children are perfectly predictable. On the other hand, we argued already that it can be seen to represent maximal movement.

Inequality aversion between types

In the traditional literature on intergenerational mobility, much attention has been devoted to cases where one row of *P* stochastically dominates another. We therefore define for any matrix $P \in \Gamma$: $C_{t,r} = \sum_{i=1}^{r} p_{t,i} \quad (t = 1,..,n; r = 1,\ldots,n)$. Row *j* of matrix *P* stochastically dominates row *l* if $C_{j,r} \leq C_{l,r} \quad \forall r = 1,\ldots,n$ with at least one inequality strict. In some cases the domain of transition matrices has been restricted to Δ, the set of so-called monotone matrices (see, e.g., Conlisk, 1989, 1990 and Dardanoni, 1993, 1995):

$$\Delta = \{P \mid C_{i+1,r} \leq C_{i,r} \forall r = 1,\ldots,n \text{ and } \forall i = 1,\ldots,n-1\}$$

A monotone matrix is a matrix where each row is stochastically dominated by the row below[9].

The concentration on this form of stochastic dominance is an indication of the fact that the literature on intergenerational mobility measurement focuses on the rows rather than the columns of P and is therefore closer in spirit to the IABT-approach of Lemma 2 than to Roemer's IAWR-approach of Lemma 1. We will therefore also focus on the former. In this between-types framework the stochastic dominance of row *j* over row *l* immediately implies that children with a parent in class *j* have better opportunities than children with a parent in class *l* for all monotonic *v*-functions.

The desirability of equalising opportunities between types has been represented in the previous section by the axiom IABTYBM. Combining this axiom with the idea of stochastic dominance, we can reformulate it easily in terms of transformations of the transition matrices:

DEOT (Desirability of Equalising Opportunity Transformations)

If $\tilde{P}$ can be obtained from P through a finite sequence of EOT, then $M[\tilde{P}] > M[P]$, where ***EOT*** is an "equalising opportunity transformation", defined as

$$\tilde{p}_{ij} = p_{ij} - \varepsilon; \tilde{p}_{ik} = p_{ik} + \varepsilon; \tilde{p}_{lj} = p_{lj} + \varepsilon; \tilde{p}_{lk} = p_{lk} - \varepsilon \quad \varepsilon > 0, \quad j < k,$$

$\tilde{C}_{i,r} \geq \tilde{C}_{l,r} \quad \forall r = 1, \ldots, n$ (with at least one of the latter inequalities strict)

A comparison between the definition of MADT in the reformulated axiom SM and the definition of EOT immediately shows

Theorem 3. **SM and DEOT are incompatible.**

The interpretation of the theorem is revealing. If we interpret social mobility as movement, any movement between income classes will increase mobility and it is reasonable to impose SM.

9 It has been argued that most empirical transition matrices are close to monotonicity. But the property is far from universal. If we look at the matrices described in Appendix 2, three of them are not monotonous: those given by Atkinson et al. (1983) for GB, by Rustichini et al. (1996) for Italy, and by Behrman and Taubman (1985) for the USA. In this last case the violations of monotonicity are particularly severe. This is mainly due to the fact that it is a large (decile) matrix: of course, distinguishing more groups will lead to more deviations from monotonicity. If each individual would be put into a cell of his own, the matrix would only be monotonous if the intergenerational process does not allow any reranking.

However, if we are interested in social mobility as a means to equalize economic opportunities, only an "equalising" movement (as defined by EOT) will increase our mobility measure. Another way of interpreting the same finding is to note that MADT-transformations are a subclass of the class of transformations defined in Atkinson and Bourguignon (1982) and that therefore the following axiom is a straightforward extension of SM:

PCP (Preference for Children of the Poor)

If $\tilde{P}$ can be obtained from P through a finite sequence of ABT, then $M[\tilde{P}] > M[P]$, where ***ABT*** is an "Atkinson-Bourguignon-transformation", defined as

$$\tilde{p}_{ij} = p_{ij} - \varepsilon; \tilde{p}_{ik} = p_{ik} + \varepsilon; \tilde{p}_{lj} = p_{lj} + \varepsilon; \tilde{p}_{lk} = p_{lk} - \varepsilon, \quad \varepsilon > 0, \quad j < k, \quad i < l$$

Given the interpretation of the transformations it is obvious that PCP implies SM, but not vice versa. The conflict between PCP and DEOT is obvious. Both transform the matrix P in the same way, but they have a different condition on the rows where the transformation takes place. By PCP all transfers of opportunities from children of richer to children of poorer parents increase mobility. According to DEOT a transformation is desirable only if it is in favor of children with worse opportunities. A transformation in favour of the children of poor parents at the expense of children with rich parents will only equalise opportunities, if the latter group had better opportunities before the transformation. This suggests that the conflict between PCP and DEOT (and hence between SM and DEOT) will disappear if we restrict the domain of transition matrices to Δ: on that domain children of poorer parents will always have poorer opportunities.

On Δ there is no longer a conflict between SM and DEOT because any MADT (and even any ABT) will be equalising. Nor is there a conflict between ANT* and SM: in fact, the anonymity axiom cannot be meaningfully applied on the domain Δ since any permutation of the rows of $P \in \Delta$ will yield a new mobility matrix outside Δ. It then stands to reason that the Shorrocks-conflict between SM and PM also will disappear for mobility matrices in Δ:

Lemma 4. **SM and PM are compatible on Δ.**

However, we do not feel that domain restrictions are an adequate answer to a conflict between different attractive axioms[10]. It may be true that non-monotone matrices are rare, but typically our sharpest intuitions involve the comparison of extreme cases. The exceptional character of the cases is not in itself a reason to throw the intuitions overboard. On the contrary, they show clearly the implications of the choice of a specific mobility measure. In this case there is a basic conflict between two approaches to social mobility: on the one hand mobility as movement, on the other hand mobility as more equal opportunities. In the former approach axiom SM is perfectly meaningful but PM is not. In the latter approach, PM seems crucial, but SM does not make too much sense. On the other hand, ANT* and DEOT (for the between-types approach) seem indispensable for a measure of equality of opportunities.

Let us therefore now take a look at the different mobility indices proposed in the statistical and sociological literature, some of which were already introduced in Table 1. Which mobility measures satisfy which axioms?[11] The details of definitions and derivations can be found in Appendix 1. Table 2 gives an overview of the results. From this extensive, though not exhaustive overview, we can conclude that all measures satisfy I, some satisfy SM, few satisfy PM and ANT* and none satisfies DEOT. To measure "mobility as movement" there is some choice: many measures satisfy SM. For those who want to analyse transition matrices from the point of view of equal opportunities, however, the existing literature does not contain an attractive index. The best choice seems to be $M_f[P]$ and for obvious reasons: it depends directly upon the deviation of P from P^M. However $M_f[P]$ does not satisfy DEOT.

Table 2 about here

This negative result is not really surprising since the literature has focused on mobility as movement. At the same time, many people are interested in intergenerational links mainly

10 To resolve the conflict between PM and SM, Shorrocks (1978) also suggests to "exclude those matrices which, by any stretch of the imagination, are unlikely to arise in practice". He therefore concentrates on matrices with a quasi-maximal diagonal and shows that PM and SM are no longer incompatible for this class of matrices. P has a quasi-maximal diagonal when there exist positive $\mu_1, \ldots, \mu_n$ such that $\mu_i p_{ii} \geq \mu_j p_{ij} \forall i, j$.

11 See Shorrocks (1993) for a similar exercise.

because they would like to see more equal opportunities. This basic link between "concern for equality of opportunities" and "intergenerational mobility" is not captured by any of the current indices. Therefore, those who look at transition matrices from this point of view can be seriously misguided if they use any of the existing indices. In the next section we will illustrate how the insights from section 2 lead directly to some easy measures of "equality of opportunities" in the context of transition matrices.

4 Evaluation of transition matrices in terms of equality of opportunities

Application of the results of Lemmas 1 and 2 to transition matrices will yield immediately an operational criterion if we are willing to choose specific functional forms. We suggest some specific choices in section 4.1. In section 4.2 we illustrate how our proposed indices work for the evaluation of the empirical matrices from Table 1 and Appendix 2.

4.1 A concrete proposal

Let us first look at Roemer's proposal in Lemma 1. We argued already that there are good reasons to impose ANT in the within-column approach too, i.e. to impose $w_{t(\rho)}[.] = w_{\rho}[.] \quad \forall t$. We further assume that the evaluation function is not dependent on ρ. This is less restrictive than it may seem at first sight. Since we identify the level of effort on the basis of the income level reached (the higher the income level, the larger the effort), the effect of differences in effort is indistinguishable from the effect of income and can therefore be captured by the specification of the valuation function. Bringing these assumptions together we write

$$S[Y] = \sum_{\rho=1}^{m} \min_{t} \eta[y_{t(\rho)}]$$

where η[.] is increasing and concave. The well known iso-elastic specification is an obvious choice for η[.]. If we work with a transition matrix the *m* effort-levels can be operationalised as the percentiles of the income distributions of the different types. Denoting these percentiles by *z*, we get

$$S_R[Y] = \sum_{z=1}^{100} \min_{t} \eta[y_{tz}]$$

where S_R stands for evaluation function according to Roemer (focusing on degree of Responsibility).

The alternative proposal in Lemma 2 focuses on the evaluation of the rows. Again assuming that the effect of effort differences is conflated with the effect of income differences, the evaluation function from Lemma 2 reduces to

$$S[Y] = \sum_{t=1}^{n} \psi\left[\sum_{\rho=1}^{m} \eta[y_{t(\rho)}]\right]$$

or, applied to transition matrices,

$$S_T[Y] = \sum_{t=1}^{n} \psi\left[\sum_{j=1}^{n} p_{tj}\eta[y_j]\right]$$

where S_T stands for evaluation function focusing on the opportunities of different Types. The income y_j is the income of the j-th quantile in the income distribution of the children. Again we could use the iso-elastic form for the functions ψ and η. A special case with extreme inequality aversion is

$$S_{LEX}[Y] = \Lambda\left[\sum_{j=1}^{n} p_{1j}\eta[y_j], \ldots, \sum_{j=1}^{n} p_{nj}\eta[y_j]\right]$$

where Λ is the leximin-ordering.

Figure 2 helps clarifying the interpretation of these valuation functions with a three-group example. The upper part of the figure shows what can be called the opportunity set of a given type *i*. Cumulative frequencies are put on the horizontal axis. They represent the effort level. The vertical axis gives the evaluation of the incomes of the three quantiles: the specific numbers will depend on the specification of η. The shaded area gives the surface of the opportunity set of type *i*. It is now clear how to interpret S_T: it is the sum of concave transformations of these surfaces. If we choose S_{LEX}, the ranking of social states will be based only on the surface of the smallest opportunity set. The alternative criterion S_R does not consider the surfaces of the opportunity sets of the different types, but instead takes the intersection of all the sets[12]. This is illustrated in the lower part of Figure 2, where we have brought together two opportunity sets: the heavy line is the intersection as measured by S_R.

Figure 2 about here

It is important to realise that the measurement of opportunities requires more than just the information in the transition matrix. Both for S_R and for S_T we also need information about the income levels of the different quantiles. This is immediately obvious from Figure 2, but the intuition can perhaps be strengthened by considering a specific example. Suppose we have to compare two situations with the same transition matrix: assume it is the identity matrix in both cases. Now suppose that in the first situation the income levels associated with the different quantiles for the children are very wide apart, while in the second situation these income levels are virtually the same. Then, although the transition matrices are identical, it is obvious that opportunities are more unequally distributed in the first than in the second situation.

Figure 2 reveals another interesting insight: S_{LEX} and S_R coincide on the domain Δ of monotone transition matrices. Indeed, on Δ, the smallest opportunity set is at the same time the intersection of all the opportunity sets. We noted already that monotonicity is a far from universal characteristic of transition matrices, however.

What we have proposed until now is the general specification of a social evaluation function embodying among other things the monotonicity condition WRI (or SRI). Once we have chosen specific functional forms for the functions $\eta[.]$ in S_T and S_R, and for $\psi[.]$ in S_T, we can derive from this social valuation function a measure of "inequality of opportunities" by the usual procedure of defining an equally distributed equivalent measure of opportunities. In the next section we will illustrate this for S_T.

12 The interpretation of S_T in terms of the surface of the opportunity sets and of S_R as the intersection of the opportunity sets is due to Marc Fleurbaey.

4.2 An empirical example

To implement S_T, we choose the iso-elastic specification for η[.] and for ψ[.], resulting in

$$S_T[Y] = \sum_{t=1}^{n} \frac{O_t^{1-\alpha} - 1}{1-\alpha} \qquad \alpha \geq 0 \tag{1}$$

where

$$O_t = \sum_{j=1}^{n} p_{tj} \frac{y_j^{1-\varepsilon} - 1}{1-\varepsilon} \qquad \varepsilon \geq 0 \tag{2}$$

We write "average opportunities" as $\overline{O} = \frac{1}{n}\sum_{t=1}^{n} O_t$ and using (1) we define the "equally distributed equivalent" level of opportunities O^E implicitly by

$$\sum_{t=1}^{n} \frac{O_t^{1-\alpha} - 1}{1-\alpha} = \sum_{t=1}^{n} \frac{(O^E)^{1-\alpha} - 1}{1-\alpha}$$

Bringing all these elements together, we compute an "index of inequality of opportunities" as

$$I^{OPP} = 1 - \frac{O^E}{\overline{O}}$$

We can then rank different situations on the basis of

$$O^E = \overline{O}(1 - I^{OPP}) \tag{3}$$

where we see the usual decomposition in a "level" component (average opportunities) and an "inequality component".

While proportional changes in the income vector keep the within-row (ethically justified) inequality constant, and proportional changes in opportunities keep I^{OPP} constant, the effect of proportional changes in incomes on I^{OPP} is less transparent. To avoid complications with different exchange rates or different price levels, we will consider in our empirical illustration the degenerate case

$$\eta[y] = y$$

such that the opportunities of type *i* are given simply by the average income for the children of that type. This makes I^{OPP} invariant for proportional changes in all incomes. This also allows us to normalise the opportunities so that the value for the children of the poorest type is equal to 1. Following this procedure we get the values in Table 3 for the opportunities of the quartiles in those transition matrices for which we could compute the marginal distributions (see Appendix 2). Of course the methods used for the construction of these mobility matrices are not always comparable and we have made some extremely strong assumptions to get at the numbers in Table 3. The present analyis is only meant to be illustrative.

Table 3 about here

From Table 3 it is evident that opportunities are most equally distributed in Italy. This is confirmed in the upper part of Figure 3, where we compare the values of I^{OPP} for different values of α. Using equation (3) it can be calculated that for high values of α, eliminating all inequality of opportunity would be equivalent to an increase in average opportunities of 13.6% in the U.S. (I^{OPP}=0.12) and of 4.4% in Italy (I^{OPP}=0.042). Comparison is not always that straightforward, as the comparison of D(GB) and Z(US) in the lower part of Figure 3 shows. For low values of α, opportunities in Britain are more unequally distributed than in the U.S. For high values, the picture changes. This reflects of course the pattern in Table 3: as α increases, the larger inequality at the bottom end of the US distribution becomes more and more important for the value of the inequality index. It is worthwhile comparing these results with the rankings obtained with the traditional mobility indices (Table 1). It turns out that integrating information about the marginal distributions in the evaluative exercise and specifying equality of opportunity in a consistent way have important consequences for the results.

Figure 3 about here

5 Conclusion

One of the main reasons to be interested in intergenerational mobility is the concern with equality of opportunity for children of different descent. Researchers that are interested in an evaluation of matrices of transition from this perspective have to be careful in at least two respects. First, the mobility measures proposed in the literature are not attractive to measure equality of opportunity. Axioms like Shorrocks' monotonicity condition may capture in an adequate way the idea of mobility as movement, but their relevance for the measurement of equality of opportunity is restricted to the class of monotone transition matrices. Secondly, the transition matrix in itself does not contain sufficient information to obtain a complete ranking of alternative states of the world. Just like in the Atkinson (1981) framework, we also need information on the marginal distribution of the incomes of both generations.

Recent developments in the theory of equal opportunities suggest two alternative approaches to measure the degree of equality of opportunities captured in a transition matrix. One starts from Roemer (1996)'s idea to aim for equal incomes for all children at the same effort level and concentrates on the columns of the transition matrix. The other one elaborates Van de gaer (1993)'s idea to measure the "opportunities" of children of different descent by using information from the rows of that matrix. Both approaches are basically incompatible but both capture important intuitions concerning equality of opportunity and can be easily operationalised. We have illustrated this with some empirical examples.

Our approach underlines the need to think carefully about the normative implications of different mobility measures. The theory of equal opportunities offers a promising starting point for the development of new and interesting tools for policy analysis. More work is needed to get a better insight into the empirical and theoretical implications of these new tools in an intergenerational context.

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APPENDICES

Appendix 1: The Axioms and Measures of Mobility

The axioms I, ANT*, PM and SM can be checked fairly easily. We will do this in the first part of this appendix, where the measures and their rationale are discussed. DEOT is more difficult to verify. For the measures for which it is not easy to establish whether DEOT is satisfied, we will check DEOT in the second section with the help of a numerical example.

1.1. Measures, I, ANT, PM and SM*

1.1.A. Statistical Measures of Mobility

(1) Measures of generational dependence.

Spectral decomposition of the matrix P yields $P=\sum_{r=1}^{n}\lambda_r A_r$ where λ_r is the r-th eigenvalue of P and $\{A_r\}$ is the corresponding spectral set. $A_r A_s=0$ if $r\neq s$, $A_r A_r=A_r$ and $\sum_{r=1}^{n}A_r=I$. The T-th period transitionmatrix which, by definition, contains the probability that a dynasty will be in state j after T periods, given that the its founding father started in state i, is given by $P^T=\sum_{r=1}^{n}(\lambda_r)^T A_r$. The largest eigenvalue of P is equal to 1 because of the stochastic nature of P so that

$$P=\iota\pi'+\sum_{r=2}^{n}\lambda_r A_r \quad \text{and} \quad P^T=\iota\pi'+\sum_{r=2}^{n}(\lambda_r)^T A_r$$

where π is the equilibrium probability vector: $\pi=\pi P$. Since the Markov chain is assumed to be regular, π is unique. If $P\neq I$, then the other eigenvalues will be smaller than 1 in absolute value and therefore $\lim_{t\to\infty}P^T=\iota\pi'$. They will determine the speed at which the transition matrix convergences to the perfectly mobile matrix, which is the situation without generational dependence. If $P=I$, all eigenvalues will be equal to 1 and we have complete generational dependence.

a) Asymptotically, the second largest eigenvalue determines this speed of convergence. Hence its absolute value provides an indication of th extend to which the father's class determines the incomes of the future generations. $M_\lambda[P]=1-|\lambda_2[P]|$ can thus be used as a measure of mobility.

The product of the eigenvalues of P is equal to the determinant of P. Inter changing rows of P does not change the absolute value of the determinant and hence the absolute value of the product of the eigenvalues. It does, however, change the trace of the matrix which is equal to the sum of the eigenvalues. The eigenvalues change and $M_\lambda[P]$ will not satisfy ANT*. Under PM, $P=\iota\pi'$, and from the introductory discussion to this section it follows that $\lambda_i = 0 \quad \forall i \geq 2$. Hence PM is always met. Since $\lambda_i = 1 \quad \forall i$ if and only if $P=I$, I is always met.

Shorrocks (1978) proposed to use $M_h[P] = e^{-h}$ where $h = -(Log[2])/(Log|\lambda_2|)$ which can be interpreted as the asymptotic half life of the chain. Since h depends on λ_2, this measure inherits all the properties of $M_\lambda[P_\varepsilon]$.

b) The arithmetic mean of the eigenvalues, $(1/(n-1))\sum_{r=2}^{n}\lambda_r$, can be used as the basis for a measure of immobility. Since the sum of the eigenvalues of P is equal to the trace of P, $M_t[P] = (n - Tr[P])/(n-1)$ can be used as a measure of mobility. The transformations of the probabilities that are allowed by SM increase the trace of a matrix, and therefore $M_t[P]$ satisfies SM. Consequently the measure violates PM and ANT*. I is satisfied.

These conclusions are valid for many sociological mobility indices (see Boudon (1973) for their definition). For the class of mobility matrices considered in this paper, $M_t[P]$ is equal to the generalized Yasuda mobility index and a variation of the Matras index. The measure is also proportional to the generalized Boudon index. Hence all these measures have the same properties. There also exist measures which depend only on the diagonal elements of *P*. This is the case for any judgment based upon the *n* values of the Glass index or their inverses, the *n* values of the Prais index. Also these measures[1] exhibit the same axiomatic properties as $M_t[P]$.

c) The absolute value of the geometric mean of the eigenvalues, $|\lambda_2\lambda_3\ldots\lambda_n|^{1/n-1}$, is another possible immobility measure. Since the product of all the eigenvalues equals the determinant of the matrix, we have that $M_d[P] = 1 - |P|^{1/n-1}$.

Changing rows in a matrix does not change the absolute value of its determinant. Therefore this measure satisfies ANT* always and cannot satisfy SM. $|P^M| = 0$ so that P^M corresponds maximal mobility. However, as soon as any two rows of the matrix P are equal, $|P| = 0$ such that PM is not satisfied. The absolute value of the product of the eigenvalues is maximal if all λ_i are equal to 1. Hence I is met.

d) It is also possible to define measures which directly depend upon the deviation of P from $i\pi'$. One such measure is $M_f[P] = 1 - (1/n^2)\sum_{j=1}^{n}\sum_{i=1}^{n}\left|\frac{p_{ij} - \pi_j}{\pi_j}\right|$.

The measure always satisfies PM, and, because switching rows in the matrix P only provokes the same permutation in π, $(\pi = \pi P)$, $M_f[P]$ always satisfies ANT* and cannot satisfy SM. I is satisfied.

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[1] The dependence of many sociological measures upon the diagonal elements of the matrix of transition only is probably due to the difficulty in ordering sociological classes such as, e.g., occupations.

(2) Measures of Movement

a) The expected proportion of family lines changing class from one generation to the next in steady state is given by $M_I[P]=\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\pi_i p_{ij}=1-\sum_{i=1}^{n}\pi_i p_{ii}$. Again, Shorrocks type transformations of P increase the diagonal elements of P, so that the measure satisfies SM, and, as a consequence, this measure cannot satisfy PM nor ANT*. I is evidently met. Note also that all transitions are treated alike, irrespective of the width of the transition (assuming states can be ordered). This was Bartholomew's (1982) motivation for the next measure

b) When we weight each class transition by the number of class boundaries which have been crossed, we get Bartholomew's measure of mobility $M_B[P]=(1/(n-1))\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_i p_{ij}|i-j|$. Again, this measure satisfies SM and does not satisfy PM nor ANT*. I is satisfied.

c) Conlisk (1990) has proposed two measures of mobility. The first one, Conlisk 's D-criterion, defines an incomplete ordering of monotone matrices of transition and is based on the degree of monotonicity of the matrices to be compared. Indeed, P^* is considered to be more mobile than P if $D[P^*]<D[P]$ where $D[P]$ has as *i,n-l* th element $C_{i,n-l}$. $D[P^*]$ is similarly defined. Both are square matrices of dimension n-1, the inequality sign has to hold for all corresponding elements of the matrices. There does not exist any matrix P such that $D[I]<D[P]$, nor does there exists any monotone matrix P,, P^M such that $D[P]<D[P^M]$. In that sense I and P^M are satisfied. Changing rows in a transition matrix affects $D[P]$ and the criterion does not meet ANT*. The transformations permitted by SM affect different rows of $D[P]$ differently, so SM does not hold true.
Let N be the matrix containing n_{ij}, the expected number of periods that it takes the chain to get from state i to state j, as elements. Then $\pi' N\pi$ has been suggested as an immobility measure (Conlisk, 1990). Let then $M_m[P]=1/\pi' N\pi$ be a measure of mobility. We have that $M_m[P_\varepsilon]=1/(1+(1/2\varepsilon))$, showing that the measure is increasing in ε, and as a consequence cannot satisfy PM nor ANT*. I is satisfied. Note that this measure satisfies the D-criterion mentioned above and so does not generally satisfy SM.

1.1.B. Measures based on the equalization of dynastic incomes

Originally, these measures were formulated to judge transition matrices on the basis of the extent to which income mobility led to an increased equality of total revenues over a longer time span for an individual. In principle these measures can also be applied to dynasties, however. When $P=I$, no equalization takes place, and, as a consequence, there will be no mobility. Therefore these criteria satisfy I.

Conlisk (1989) proposes to rank the matrices of transition on the basis of the equalization of the present value of expected incomes that result from repeated application of this matrix. To see that PP (and hence ANT*) nor PM are satisfied, consider the following example. Let $y^1 = \ldots = y^i = \ldots y^\tau = y$. Then the present value of a dynasty 's income over τ generations becomes $Y[P] = y + \rho P y + \ldots + \rho^\tau P^\tau y$. Since the properties have to hold for all possible y vectors and for all possible values of τ and ρ, let $\rho = 1$, $\tau = 2$ and $y_i = c - y_{n-i} \forall i$. Then $Y[P]$ will be completely equalized for the matrix P^A which has zeros everywhere except on the non-main diagonal where it has ones. This contradicts PM and PP since the 'best' matrix is given by P^A. P^A is a permutation of I and therefore ANT* is not satisfied by Conlisk's criterion. The criterion does not satisfy SM in general either. For, take the same example as above with $\tau = 1$. If $\exists i < j: Y_i[P] > Y_j[P]$, SM is not met. In that case, a transformation T_{ij} makes a relatively bad outcome more likely for dynasty i in the next period. This lowers $Y_i[P]$. At the same time a good outcome becomes more likely for dynasty j, increasing $Y_j[P]$. Therefore the transformation has led to a transfer which equalized the present value of the expected lifetime incomes, and, consequently increased mobility, which contradicts SM.

The criterion proposed by Chakravarty et al. (1985) is similar in spirit to Conlisk's criterion. They propose to rank social states on the basis of the decrease in inequality measures, caused by mobility. These measures have the same properties as Conlisk's: they do not satisfy PM, ANT* nor SM.[2]

1.1.C. Measures that require information on the marginal distributions

(1) Measures based on regression analysis

These measures have been very popular in econometric research. They start from a very traditional economic specification: $\ln[y^i_{t+1}] = \alpha_t + \beta_t \ln[y^i_t] + \varepsilon^i$. The estimate of the coefficient β, $\hat{\beta}$ is then interpreted as a coefficient of immobility. Hence, $M_{\hat{\beta}} = 1 - \hat{\beta}$ is a measure of mobility. Since $\hat{\beta} = \frac{Cov\{\ln[y_t], \ln[y_{t+1}]\}}{Var\{(y_t)^2\}}$, it is easy to establish that this measure satisfies SM. SM type of transformations decrease the covariance term, thereby decreasing $\hat{\beta}$ and increasing mobility. As a result, the measure satisfies I, but cannot satisfy ANT*, PM, nor DEOT.

The Hart measure of mobility, extensively discussed in Shorrocks (1993), is defined as $M_H = 1 - r[\ln[y_t], \ln[y_{t+1}]]$, where $r[\ln[y_t], \ln[y_{t+1}]]$ is the correlation coefficient between incomes of different generations. Provided that the covariance between

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[2] To verify this claim, consider social welfare functions which are additively separable over individuals. Reformulate the problem in terms of matrices of transition and reconstruct the counter-example given above for Conlisk.

$\ln[y_t^i]$ and ε^i is zero, we have that $M_H = 1 - \hat{\beta}\frac{\sigma[\ln[y_t]]}{\sigma[\ln[y_{t+1}]]}$ where $\sigma[\cdot]$ is the sample standard deviation. For given marginal distributions, M_H varies inversely with $\hat{\beta}$. Hence, it satisfies SM and therefore I, but cannot satisfy PM, ANT* nor DEOT.

(2) Measures of Distributional Change

These measures, which we have labeled M_{DC}, have been proposed to evaluate changes of frequency distributions and are therefore suited to analyze the problem of intergenerational mobility. Their properties come out most clearly in the axiomatic treatment by Cowell (1985). They satisfy SM and hence I, but not PM, ANT* nor DEOT.

1.2. Measures and DEOT

Consider the following transition matrix:

$$A[a] = \begin{bmatrix} 0.5 & 0.25+a & 0.25-a \\ 0.25 & 0.5-a & 0.25+a \\ 0.25 & 0.25 & 0.5 \end{bmatrix} \text{ where } -0.25<a<0.25$$

This matrix is clearly monotonous. Clearly Conlisk's D-criterion cannot generally satisfy DEOT because increasing a increases monotonicity between the first and second row, but decreases monotonicity between the second and third row. For the other measures, DEOT requires that, for $a_1 > 0$, $M[A[0]] > M[A[a_1]]$. However, the measures of mobility provide the following values for mobility:

measure	$M[A[0]]$	$M[A[0.1]]$
$M_\lambda[P] = 1 - \lvert\lambda_2[P]\rvert$, $M_h[P] = e^{-h}$	0.25	0.25
$M_t[P] = (n - Tr[P])/(n-1)$	0.75	0.8
$M_d[P] = 1 - \lvert P\rvert^{1/(n-1)}$	0.75	0.806
$M_f[P] = 1 - \left(1/n^2\right)\sum_{j=1}^{n}\sum_{i=1}^{n}\left\lvert\frac{p_{ij}-\pi_j}{\pi_j}\right\rvert$	0.667	0.711
$M_l[P] = 1 - \sum_{i=1}^{3}\left(1/3\right)p_{ii}$	0.5	0.533
$M_B[P] = (n-1)\sum_{i=1}^{n}\sum_{j=1}^{n}\left(p_{ij}/n\right)\lvert i-j\rvert$	1/3	1/3
$M_m[P]$	0.272	0.285

Appendix 2: Empirical Implementation

The following table provides a short description of the mobility matrices used in the construction of table 1 in section 3 and the empirical example of section 4.

Source Country	name data set	year (F/S)	N. of Obs	N. Of Class	income variable	info on marg. distrib.
Atkinson (1983) GB	Rowntree follow-up	I: 1950 I:1975-8	374	5	age-adjusted hourly earnings	Some
Behrman and Taubman (1985) US	NAS-NRS Twin sample	B:1917-27 I:1977-81	1025	10	experience adjusted yearly earnings	None
Zimmerman (1992) US	NLS panel	I:1965 I:1981	278	4	log yearly earnings	None
Rustichini, Ichino and Checci (1996)	It:INMS	I:1985	1681	4	median occupatio-nal income	Some
US/It	US: PSID	I:1974 I:1990	1050	4	median occupatio-nal income	Some
Dearden, Machin and Reed (1997) GB	NCDS panel	I:1974 B:1958/I: 1991	1565	4	predicted permanente arnings	None

I: the year the interview was taken
B: the year of birth

Few studies report information on the marginal distributions of fathers' or sons' incomes. A notable exception is Rustichini et al (1996). We used his data to fit a lognormal distribution of incomes. Due to the fact that his data are based on median occupational incomes, we get an underestimation of inequality. Atkinson (1983) also provides some information on the distribution of hourly earnings in Great Britain. We used his data to fit a lognormal distribution of hourly earnings for Great Britain.

Rustichini et al. (1996)'s article contains 8 points of the distribution of their index (octiles) of occupational incomes. These incomes were normalised such that the minimal value of occupational income equals 100. The estimation procedure fits a cumulative lognormal distribution with origin equal to 100 to these eight data points. The parameter values of the lognormal distribution are those which minimize the sum of squared residuals. In the present context, this is not a maximum likelihood procedure, but a distance minimization procedure. The same procedure was applied to the 4 data points mentioned in Atkinson (1983). With some imagination the origin of the lognormal distribution was put equal to 1.0. This yielded the following results:

R (US): LN{4.687, 0.735}, origin 100
R (It): LN{3.321,0.724}, origin 100
A (GB): LN{-0.472,0.890}, origin 1.00

The estimated lognormal distributions were used to calculate the mean income per quartile of the childrens' income distribution. We obtained the following values:

	bottom	25-50	50-75	Top
R (It)	111.7	122.1	135.4	174.8
R (US)	145.2	186.4	239.2	397.9
A (GB)	1.218	1.474	1.845	3.170

Appendix 3: Proofs of Lemmas 1, 2 and 3.

Lemma 1

The proof follows directly from combining the different axioms in the following way:

WRI+SER $\Leftrightarrow S[Y] = F\left[u_1\left[y_{.(1)}\right], u_2\left[y_{.(2)}\right], \ldots, u_m\left[y_{.(m)}\right]\right]$

with $S[\;]$ and $F[\;]$ non decreasing

EIAWR $\Leftrightarrow u_r\left[y_{.(r)}\right] = \min_t u_{tr}\left[y_{t(r)}\right]$

ADBR $\Leftrightarrow S[Y] = \Omega\left[\sum_{r=1}^{m} u_r\left[y_{.(r)}\right]\right]$

Lemma 2

The proof again follows directly upon combining the different axioms:

(a) WRI+SET+IABT $\Leftrightarrow S[Y] = G\left[v_1\left[y_{1(.)}\right], \ldots, v_n\left[y_{n(.)}\right]\right]$

with $G[\;]$ and $v_t[\;]$ non decreasing and $v_t[\;]$ constant sum strictly quasi-concave (theorem 1, a)

(b) ANT $\Leftrightarrow v_t\left[y_{t(.)}\right]$ does not depend upon t: $v_t\left[y_{t(.)}\right] = v\left[y_{t(.)}\right]$

AND $G[.,.,\ldots,.]$ is symmetric

(c) ADWT $\Leftrightarrow v_t\left[y_{t(.)}\right] = \Psi_t\left[\sum_{r=1}^{m} h_{t(r)}\left[y_{t(r)}\right]\right]$

(d) ADBT $\Leftrightarrow S[Y] = \Theta\left[\sum_{t=1}^{n} v_t\left[y_{t(.)}\right]\right]$

The lemma follows by introducing (c) and (d) in (a) and by taking into account that, because of ANT, the subscript *t* of the *v*- function has to be dropped, which means that the subscript *t* in the h -function has to be dropped.

Lemma 3

IABTY compares income vectors which are such that

$v_t\left[\tilde{y}_{t(.)}\right] = v_t\left[y_{t(.)}\right] \quad \forall t \neq k, l$

AND $v_k\left[y_{k(.)}\right] < v_k\left[\tilde{y}_{k(.)}\right] \leq v_l\left[\tilde{y}_{l(.)}\right] < v_l\left[y_{l(.)}\right]$, where the strict inequalities follow from SRI.

Then, IABTY says that these vectors are to be preferred.

This *implies* that we also prefer vectors which are such that

$v_t\left[\tilde{y}_{t(.)}\right] = v_t\left[y_{t(.)}\right] \quad \forall t \neq k, l$ AND $v_k\left[y_{k(.)}\right] < v_k\left[\tilde{y}_{k(.)}\right] < v_l\left[\tilde{y}_{l(.)}\right] < v_l\left[y_{l(.)}\right]$

To prove Lemma 3, we now apply Sen's Leximin Derivation Theorem (Sen, 1986, p.1119). This theorem is applied in three steps:

(a) SRI+IABTY $\Rightarrow$ HE

Following Sen (1986, p.1116) HE, Hammond's Equity can be formally defined as follows:

Let X be the set of alternative social states and H the set of individuals.

For any $x, y \in X$, if some pair $g, h \in H$, $U_g\left[y\right] > U_g\left[x\right] > U_h\left[x\right] > U_h\left[y\right]$ and $\forall i \neq g, h, U_i\left[x\right] = U_i\left[y\right]$, then xRy

(note: we have 'implied' but not 'equivalent to' because of the strong inequality sign in the middle, and *xRy* in stead of *xPy*.)

(b) We also have that SRI $\Leftrightarrow$ P (Strong Pareto)

Following Sen (1986, p.1115, fn 61), P can be defined as:

$\forall x, y \in X, \left[\forall i\colon xR_i y \,\&\, \exists i\colon xP_i y\right] \Rightarrow xPy$ and $\left[\forall i\colon xI_i y\right] \Rightarrow xIy$

(note: it is here that SRI is needed in stead of WRI)

(c) In addition, ANT $\Leftrightarrow$ A (Anonimity)

Following Sen (1986, p.1116), A can be defined as:

If $\left\{U_i\right\}$ is a re-ordering (permutation) of $\left\{U_i^*\right\}$, then $F\left[\left\{U_i\right\}\right] = F\left[\left\{U_i^*\right\}\right]$

Therefore, SET+SRI+IABTY+ANT $\Rightarrow$ HE+P+A. The proof is completed by noting that Sen's Leximin Derivation Theorem establishes the equivalence between HE+P+A and Leximin.

<u>Table 1. Social mobility in some published transition matrices</u>

	M_λ	M_t	M_d	M_f	M_l	M_B	M_m
A (GB)	0.608	0.826	0.865	0.522	0.661	0.282	0.883
D (GB)	0.606	0.862	0.950	0.707	0.646	0.317	0.859
B (US)	0.850	0.970	0.943	0.721	0.873	0.331	0.971
Z (US)	0.665	0.870	0.947	0.710	0.652	0.314	0.856
R (US)	0.670	0.881	0.943	0.710	0.661	0.319	0.887
R (It)	0.660	0.920	0.918	0.684	0.690	0.328	0.904

See appendix 1 for the definition of the mobility indices and appendix 2 for information concerning the transition matrices.

Table 2. Axiomatic analysis of mobility measures

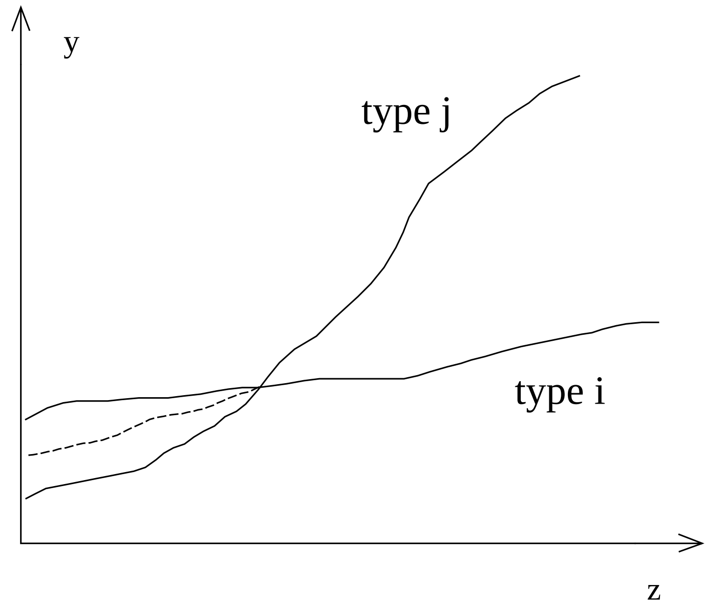
measure	I	SM	PM	ANT*	DEOT
$M_\lambda[P]$	y	n	y	n	n
$M_t[P]$	y	y	n	n	n
$M_d[P]$	y	n	n	y	n
$M_f[P]$	y	n	y	y	n
$M_l[P]$	y	y	n	n	n
$M_B[P]$	y	y	n	n	n
$M_m[P]$	y	n	n	n	n
Conlisk's D	y	n	y	n	n
Chakravarty	y	n	n	n	n
$M_{\hat{\beta}}$	y	y	n	n	n
M_H	y	y	n	n	n
M_{DC}	y	y	n	n	n

See appendix 1 for the definition of the mobility indices.

Table 3. Opportunities of children in some published transition matrices

	Children of bottom	Children of 25-50	Children of 50-75	Children of top
Z (US)	1.000	1.160	1.322	1.360
R (US)	1.000	1.033	1.188	1.378
R (It)	1.000	1.000	1.053	1.156
D (GB)	1.000	1.020	1.086	1.441

Figure 1. The conflict between IAWR and IABT



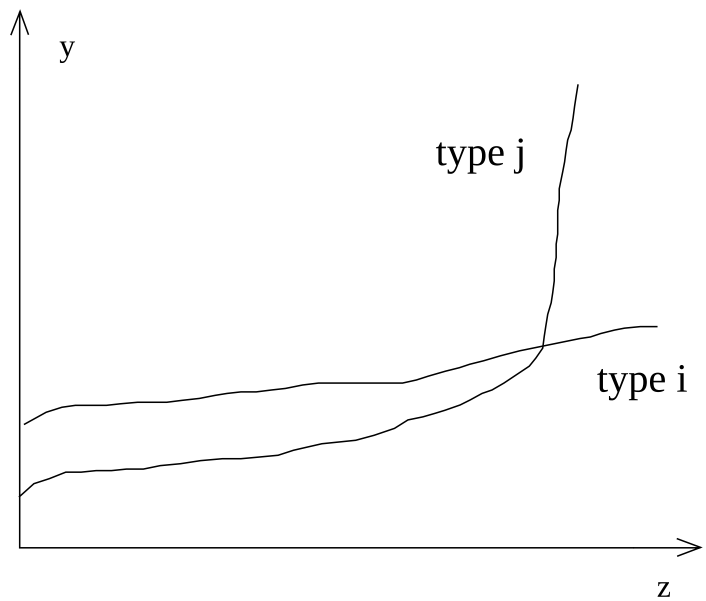


Figure 2. Transition matrices and opportunity sets