STATISTICAL INFERENCE FOR 2 MEASURES OF INEQUALITY

WHEN INCOMES ARE CORRELATED

by

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ABSTRACT

This paper derives the asymptotic normality of the distribution of differences in the Atkinson/Kolm and generalised entropy inequality measures when incomes are correlated. We illustrate the procedure by calculating the difference in inequality before and after tax in Ireland. Our empirical results suggest that the positive correlation between incomes before and after taxes substantially reduces the variance of the estimated difference in inequality.

Keywords: Inequality; Redistribution; Distribution Free Statistical Inference

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1. INTRODUCTION

There are two ways of doing statistical inference with measures of inequality. One can use a bootstrap method to calculate bootstrapped standard deviations of inequality measures, or of a function of inequality measures –see, e.g. Mills and Zandvakili (1997). Alternatively, one can try to establish the large sample distribution of the inequality measure. Cowell (1989) and Thistle (1990) have established the large sample properties of the Atkinson/Kolm inequality measure and of the generalised entropy measure of inequality[1]. These results are very important because they allow us to perform distribution free statistical inference. When we want to do statistical inference on two samples were incomes are correlated, the framework has to be extended, however. That is the object of the present paper.

Correlated samples arise naturally in many economic applications. We might be interested in a comparison of the inequality of income before and after tax. Kiefer's (1985) progressivity index is the difference in income inequality before and after taxes. Or we could have a panel data set, and look at the evolution of the distribution of income over time. A concern with correlated incomes is not new. Davidson and Duclos (1997) and Duclos (1997) have recently extended the existing literature about the asymptotic distribution of general functions of quantile-based estimators to deal with precisely this problem. Their results can be used to do distribution free statistical inference with the Gini index when incomes are correlated. The problem of statistical inference with the Atkinson /Kolm and generalised entropy measures of inequality seems to us an important complement to their analysis. Our results allow us to address questions like whether an observed decrease in these inequality measures after tax is likely to be due to statistical variation or not.

The next section contains the main results of this paper. The third section provides an empirical illustration that compares inequality before and after tax in Ireland in 1994. Section four concludes.

[1] Cowell (1989) derives the large sample distribution of the Gini coefficient too. For a discussion of these measures and their properties, see, e.g., Cowell (1995).

2. FRAMEWORK

2.1. Preliminaries

Let y and y' be two positive[2] and correlated random variables with joint distribution $F_J[y, y']$ and marginal distributions $F[y]$ and $F'[y']$. Then the α-th moment of the distribution of $F[y]$ is defined as

$$\mu_\alpha[y] = \int_0^\infty y^\alpha \, dF[y] \qquad \alpha \in \Re$$

if the integral exists. $\mu_\alpha[y']$ is similarly defined.

Suppose we have a random sample $\{(y_1, y'_1) \;,\ldots,\; (y_i, y'_i) \;,\ldots,\; (y_n, y'_n)\}$ from $F_J[y, y']$. The natural estimator of $\mu_\alpha[y]$ is given by $m_\alpha[y] = \frac{1}{n}\sum_{i=1}^{n}(y_i)^\alpha$. The following lemma will be essential to derive the asymptotic properties of differences in inequality measures when incomes are correlated.

Lemma:

$$\left[\sqrt{n}(m_\alpha[y] - \mu_\alpha[y]) \quad \sqrt{n}(m_{\alpha'}[y] - \mu_{\alpha'}[y]) \quad \sqrt{n}(m_\alpha[y'] - \mu_\alpha[y']) \quad \sqrt{n}(m_{\alpha'}[y'] - \mu_{\alpha'}[y'])\right]$$

is asymptotically multivariate normally distributed with mean zero and variance-covariance matrix

$$\Omega_{\alpha\alpha'}[y, y'] = \begin{bmatrix} \omega_{\alpha\alpha}[y, y] & \omega_{\alpha'\alpha}[y, y] & \omega_{\alpha\alpha}[y', y] & \omega_{\alpha'\alpha}[y', y] \\ \omega_{\alpha\alpha'}[y, y] & \omega_{\alpha'\alpha'}[y, y] & \omega_{\alpha\alpha'}[y', y] & \omega_{\alpha'\alpha'}[y', y] \\ \omega_{\alpha\alpha}[y, y'] & \omega_{\alpha'\alpha}[y, y'] & \omega_{\alpha\alpha}[y', y'] & \omega_{\alpha'\alpha}[y', y'] \\ \omega_{\alpha\alpha'}[y, y'] & \omega_{\alpha'\alpha'}[y, y'] & \omega_{\alpha\alpha'}[y', y'] & \omega_{\alpha'\alpha'}[y', y'] \end{bmatrix}$$

where

$$\begin{aligned} \omega_{\alpha\alpha'}[y, y'] &= E\left\{\left(y^\alpha - \mu_\alpha[y]\right)\left((y')^{\alpha'} - \mu_{\alpha'}[y']\right)\right\} \\ &= \int_0^\infty\int_0^\infty y^\alpha (y')^{\alpha'} d_y d_{y'} F_J[y, y'] - \mu_\alpha[y]\mu_{\alpha'}[y'] \end{aligned}$$

Proof: The sequence of random vectors $\left[(y_i)^\alpha \quad (y_i)^{\alpha'} \quad (y'_i)^\alpha \quad (y'_i)^{\alpha'}\right]$ i=1,…,n is independently identically distributed with mean vector $[\mu_\alpha[y] \quad \mu_{\alpha'}[y] \quad \mu_\alpha[y'] \quad \mu_{\alpha'}[y']]$ and variance-covariance matrix $\Omega_{\alpha\alpha'}[y, y']$ defined

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[2] In our empirical section we will discuss the consequences of zero incomes.

above. Then the lemma follows immediately from the multivariate central limit theorem (see, e.g., Lehmann, E.L. (1983), p.343).

This lemma establishes that $m_\alpha[y]$ is a consistent estimator of $\mu_\alpha[y]$ and that the asymptotic variance-covariance matrix of $[m_\alpha[y] \quad m_{\alpha'}[y] \quad m_{\alpha'}[y] \quad m_{\alpha'}[y']]$ is given by (1/*n*) $\Omega_{\alpha\alpha'}[y, y']$. A consistent estimator of the elements of $\Omega_{\alpha\alpha'}[y, y']$ is readily available by

$$\hat{\omega}_{\alpha\alpha'}[y, y'] = \frac{1}{n}\sum_{i=1}^{n}(y_i)^{\alpha}(y_i')^{\alpha'} - m_\alpha[y]m_{\alpha'}[y']$$

2.2. Atkinson/ Kolm Index

The Atkinson/ Kolm inequality index for the population is

$$A_\theta[y] = 1 - \frac{1}{\mu_1[y]}\left(\int_0^\infty y^\theta\, dF[y]\right)^{1/\theta} = 1 - \frac{(\mu_\theta[y])^{1/\theta}}{\mu_1[y]} \quad \text{with } -\infty<\theta\leq 1 \text{ and } \theta\neq 0$$

The natural estimator of $A_\theta[y]$ is given by

$$\hat{A}_\theta[y] = 1 - \frac{(m_\theta[y])^{1/\theta}}{m_1[y]} \quad \text{with } -\infty<\theta\leq 1 \text{ and } \theta\neq 0$$ [3]

Thistle (1990, p.726) then shows that $\hat{A}_\theta[y]$ is asymptotically normally distributed with mean $A_\theta[y]$ and asymptotic variance equal to

$$v_{\hat{A}_\theta}[y] = \frac{1}{n}\left(\left(\frac{(\mu_\theta[y])^{\frac{1-\theta}{\theta}}}{\theta\mu_1[y]}\right)^2 \omega_{\theta\theta}[y, y] - 2\left(\frac{(\mu_\theta[y])^{\frac{2-\theta}{\theta}}}{\theta(\mu_1[y])^3}\right)\omega_{\theta 1}[y, y] + \left(\frac{(\mu_\theta[y])^{\frac{1}{\theta}}}{(\mu_1[y])^2}\right)^2 \omega_{11}[y, y]\right)$$

This expression can be derived from the lemma above by using the upper left 2 by 2 block of $\Omega_{\alpha\alpha'}[y, y']$, the differentiability of $\hat{A}_\theta[y]$ and a theorem by Rao (see, e.g., Lehmann, E.L. (1983), p.344).

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[3] The cases where θ=0 and $\theta\to-\infty$ are discussed in the appendix.

For the reasons given in the introduction, it is often interesting to test whether two vectors of correlated incomes have the same level of inequality[4]. The following theorem provides the basis for doing just that:

Theorem 1: Under the null hypothesis of equality of $\hat{A}_\theta[y]$ and $\hat{A}_\theta[y']$, $\hat{A}_\theta[y]-\hat{A}_\theta[y']$ is asymptotically normally distributed with mean zero and asymptotic variance given by

$$v_{\hat{A}_\theta[y]-\hat{A}_\theta[y']}=\frac{1}{n}a_\theta[y,y']'\,\Omega_{1\theta}[y,y']\,a_\theta[y,y']$$

$$\text{where } a_\theta[y,y']=\left[\frac{(\mu_\theta[y])^{\frac{1}{\theta}}}{(\mu_1[y])^2}\quad -\frac{(\mu_\theta[y])^{\frac{1}{\theta}-1}}{\theta\mu_1[y]}\quad -\frac{(\mu_\theta[y'])^{\frac{1}{\theta}}}{(\mu_1[y'])^2}\quad \frac{(\mu_\theta[y'])^{\frac{1}{\theta}-1}}{\theta\mu_1[y']}\right]$$

Proof: follows from the lemma, the differentiability of $\hat{A}_\theta[y]$ and $\hat{A}_\theta[y']$ and a theorem by Rao (see, e.g., Lehmann, E.L. (1983), p.344).

2.3. Generalised entropy measure of inequality

Defining $k=\dfrac{1}{\theta^2-\theta}$, the generalised entropy measure of inequality in the population is

$$G_\theta[y]=k\int_0^\infty\left(\left(\frac{y}{\mu_1[y]}\right)^\theta-1\right)dF[y]=k\left(\frac{\mu_\theta[y]}{(\mu_1[y])^\theta}-1\right)\quad \text{with } \theta\neq 0 \text{ and } \theta\neq 1$$

The natural estimator of $G_\theta[y]$ is given by

$$\hat{G}_\theta[y]=k\left(\frac{m_\theta[y]}{(m_1[y])^\theta}-1\right) \text{ with } \theta\neq 0 \text{ and } \theta\neq 1^5$$

Cowell (1989) and Thistle (1990) prove that the estimator is asymptotically normally distributed with asymptotic variance

[4] All the theorems that follow can be easily extended to deal with the case where we want to test a restriction that takes the form of any differentiable function of the inequality measures of *y* and *y'*.
[5] The cases where $\theta=0$ and $\theta=1$ are discussed in the appendix.

$$v_{\hat{G}_\theta}[y] = \frac{1}{n} k^2 \mu_1[y]^{-2(\theta+1)} \left((\mu_1[y])^2 \omega_{\theta\theta}[y,y] - 2\theta\mu_1[y]\mu_\theta[y]\omega_{\theta 1}[y,y] + \theta^2 (\mu_\theta[y])^2 \omega_{11}[y,y] \right)$$

This expression can also be derived from the lemma above by using the upper left 2 by 2 block of $\Omega_{\alpha\alpha'}[y, y']$, the differentiability of $\hat{A}_\theta[y]$ and a theorem by Rao (see, e.g., Lehmann, E.L. (1983), p.344).

Testing whether two correlated income vectors have the same inequality as measured by the generalised entropy measure can be done on the basis of theorem 2.

Theorem 2: under the null hypothesis of equality of $\hat{G}_\theta[y]$ and $\hat{G}_\theta[y']$, $\hat{G}_\theta[y] - \hat{G}_\theta[y']$ is asymptotically normally distributed with mean zero and variance

$$v_{\hat{G}_\theta[y]-\hat{G}_\theta[y']} = \frac{1}{n} g_\theta[y,y']' \, \Omega_{1\theta}[y,y'] g_\theta[y,y']$$

where $g_\theta[y,y'] = \left[-\frac{k\theta\mu_\theta[y]}{(\mu_1[y])^{\theta+1}} \quad \frac{k}{(\mu_1[y])^\theta} \quad \frac{k\theta\mu_\theta[y']}{(\mu_1[y'])^{\theta+1}} \quad -\frac{k}{(\mu_1[y'])^\theta} \right]$

3. EMPIRICAL RESULTS

To illustrate the above analysis, we calculate measures of inequality for income including social benefits before tax and income including benefits after tax. We compute the standard errors of our inequality measures and the standard errors of the differences in inequality. The data we use come from the Irish household budget survey of 1994 that contains data on 7877 households. There are no negative incomes in our sample, but some incomes are zero: 11 before tax and 22 after tax.

The presence of zero incomes can be problematic for the calculation of our two measures of inequality[6]. A single zero observation might determine their value.

[6] The problem with zero observations is related to the problem of robustness of the estimator of inequality as analysed by Cowell and Victoria-Feser (1996). They show that, if the mean of the income distribution is known, then the generalised entropy measure is not robust if $\theta \leq 0$ in the sense that a few low observations might drive the entire result. If $\theta \geq 1$, a few very large values of incomes can have a similar effect. Moreover, if the mean of the income distribution has to be estimated on the basis of the sample, then the generalised entropy measure will never be robust to very large values of incomes.

From the equation given in the main text, it follows that $\hat{G}_q[y]$ becomes infinity if there is a zero observation and q<0. Since Log[0]=-∞, the expression for $\hat{G}_0[y]$ given in the appendix implies that he same holds true for q=0. The range of parameter values for the Atkinson Index has to be restricted in a similar way. If q<0, the Atkinson index becomes one. From the appendix, it is easy to see that the same is true for zero observations when q=0 or q→-∞. In sum: in the presence of zero incomes, q has to be strictly positive for the generalised entropy measure of inequality and q has to be between 0 and 1 or equal to 1 for the Atkinson measure of inequality. Traditionally, with an Atkinson social welfare function e=1−q is called the inequality aversion parameter. When zero incomes are present, ε∈[0,1): we can only express 'moderate inequality aversion'. We then have the following results:

Table 1: Inequality before and after taxes in Ireland, 1994

a) Atkinson inequality index

1-θ	A[y]	A[y-t]	A[y]-A[y-t]
0.20	0.0481 (0.0009)	0.0366 (0.0008)	0.0115 (0.0004)
0.40	0.0938 (0.0016)	0.0717 (0.0015)	0.0221 (0.0007)
0.50	0.1159 (0.0019)	0.0890 (0.0018)	0.0269 (0.0008)
0.60	0.1374 (0.0022)	0.1063 (0.0021)	0.0311 (0.0010)
0.80	0.1804 (0.0030)	0.1434 (0.0032)	0.0370 (0.0017)
0.99	0.3138 (0.0288)	0.3652 (0.0375)	-0.0486 (0.0265)

b) Generalised entropy index

θ	G[y]	G[y-t]	G[y]-G[y-t]
0.01	0.3797 (0.0422)	0.4580 (0.0595)	-0.0783 (0.0418)
0.25	0.2419 (0.0042)	0.1875 (0.0041)	0.0544 (0.0021)
0.50	0.2389 (0.0041)	0.1822 (0.0037)	0.0567 (0.0018)
0.75	0.2410 (0.0043)	0.1829 (0.0039)	0.0581 (0.0018)
1.25	0.2565 (0.0056)	0.1937 (0.0052)	0.0628 (0.0025)
2.00	0.3145 (0.0107)	0.2348 (0.0108)	0.0797 (0.0051)

In both panels, standard errors are given in parenthesis. The standard errors of the inequality measures are quite small, and very similar in size to the ones obtained by Mills and Zandvakili (1997)[7]. The standard errors of the difference between

[7] We refer here to their asymptotic standard errors, both bootstrapped and asymptotic, of the Theil index of inequality for the NLSY data. They perform a test for the change of inequality over time –a clear application of correlated incomes. They only calculate bootstrapped standard errors in this case, however.

inequality measures are about half the size of the standard errors of the inequality measures themselves. This is due to the huge positive correlation between incomes before and after taxes. Inequality after tax is statistically significantly smaller than inequality before tax, except when θ becomes very small. In that case, the larger number of zero incomes after tax drive the result and inequality after taxes is bigger than inequality before taxes. That difference is not statistically significant, however.

4. CONCLUSION

This paper provides a framework for the distribution free statistical inference of functions of the Atkinson and generalised entropy measures of inequality when incomes are correlated. The standard errors of these functions can be calculated relatively easily. We have illustrated the analysis with an empirical application of inequality before and after taxes. Our results suggest that the positive correlation between these incomes reduces the standard error of the difference in inequality before and after taxation substantially.

APPENDIX: SPECIAL CASES

(A) Atkinson/ Kolm Index

i) When θ=0 the index is

$$A_0[y] = 1 - \frac{Exp[\rho[y]]}{\mu_1[y]}$$

where $\rho[y] = \int_0^\infty \log[y] dF[y]$. $\rho[y]$ is a random variable, whose natural estimator is given by $r[y] = \frac{1}{n}\sum_{i=1}^{n} \log[y_i]$. The natural estimator for $A_0[y]$ is given by

$$\hat{A}_0[y] = 1 - \frac{Exp[r[y]]}{m_1[y]}$$

We can now formulate a lemma that is similar to the lemma in the main text:

Lemma A1:

$\left[\sqrt{n}(m_\alpha[y] - \mu_\alpha[y]) \quad \sqrt{n}(r[y] - \rho[y]) \quad \sqrt{n}(m_\alpha[y'] - \mu_\alpha[y']) \quad \sqrt{n}(r[y'] - \rho[y'])\right]$ is asymptotically multivariate normally distributed with mean zero and variance-covariance matrix

$$\Psi_{\alpha r}[y, y'] = \begin{bmatrix} \psi_\alpha[y] & \psi_{r\alpha}[y] & \psi_{\alpha\alpha}[y', y] & \psi_{r\alpha}[y', y] \\ \psi_{\alpha r}[y] & \psi_r[y] & \psi_{\alpha r}[y', y] & \psi_{rr}[y', y] \\ \psi_{\alpha\alpha}[y, y'] & \psi_{r\alpha}[y, y'] & \psi_\alpha[y'] & \psi_{r\alpha}[y'] \\ \psi_{\alpha r}[y, y'] & \psi_{rr}[y, y'] & \psi_{\alpha r}[y'] & \psi_r[y'] \end{bmatrix}$$

where

$$\psi_\alpha[y] = \omega_{\alpha\alpha}[y, y]$$

$$\psi_{\alpha\alpha}[y, y'] = \omega_{\alpha\alpha}[y, y']$$

$$\psi_r[y] = E\{(\log[y])^2\} - (\rho[y])^2$$

$$\psi_{rr}[y, y'] = E\{\log[y]\log[y']\} - \rho[y]\rho[y']$$

$$\psi_{\alpha r}[y, y'] = E\{y^\alpha \log[y']\} - \mu_\alpha[y]\rho[y']$$

$$\psi_{\alpha r}[y] = E\{y^\alpha \log[y]\} - \mu_\alpha[y]\rho[y] = \psi_{r\alpha}[y]$$

$$\psi_{r\alpha}[y, y'] = E\{(y')^\alpha \log[y]\} - \mu_\alpha[y']\rho[y]$$

The proof of lemma A1 is similar to the proof of the lemma in the main text. Each of the elements of $\Psi_{\alpha r}[y, y']$ can be consistently estimated in the usual way. Because of Rao's theorem, $\hat{A}_0[y]$ is asymptotically normally distributed with mean $A_0[y]$ and variance

$$v_{\hat{A}_0}[y] = \frac{1}{n}\left(\left(\frac{Exp[\rho[y]]}{(\mu_1[y])^2}\right)^2 \psi_1[y] - 2\left(\frac{Exp[\rho[y]]}{(\mu_1[y])^2}\right)\left(\frac{Exp[\rho[y]]}{\mu_1[y]}\right)\psi_{1r}[y] + \left(\frac{Exp[\rho[y]]}{\mu_1[y]}\right)^2 \psi_r[y]\right)$$

In addition, we can establish theorem A1:

Theorem A1: Under the null hypothesis of equality of $\hat{A}_0[y]$ and $\hat{A}_0[y']$, $\hat{A}_0[y] - \hat{A}_0[y']$ is asymptotically normally distributed with mean zero and asymptotic variance given by

$$v_{\hat{A}_0[y]-\hat{A}_0[y']} = \frac{1}{n} a_0[y, y']' \, \Psi_{1r}[y, y'] a_0[y, y']$$

where $a_0[y, y'] = \left[\frac{Exp[\rho[y]]}{(\mu_1[y])^2} \quad -\frac{Exp[\rho[y]]}{\mu_1[y]} \quad -\frac{Exp[\rho[y']]}{(\mu_1[y'])^2} \quad +\frac{Exp[\rho[y']]}{\mu_1[y']}\right]$

ii) When $\theta \to -\infty$, the Atkinson index is

$$A_\infty[y] = 1 - \frac{F^{-1}[0]}{\mu_1[y]}$$

As pointed out by Thistle (p.727), the natural estimator for $A_\infty[y]$ is given by

$$A_\infty[y] = 1 - \frac{x_{(1)}}{\mu_1[y]}$$

where $x_{(1)}$ is the minimal order statistic, $\min\{y_i\}$. The limiting distribution of the latter depends on the population distribution, however. So does the limiting distribution of the estimator and no distribution free statistical inference is possible.

(B) Generalised Entropy

i) When θ=0, the generalised entropy index is

$$G_0[y] = \log[\mu_1[y]] - \rho[y]$$

The natural estimator of $G_0[y]$ is

$$\hat{G}_0[y] = \log[m_1[y]] - r[y]$$

This means that $\hat{G}_0[y]$ is a function of exactly the same random variables as $\hat{A}_0[y]$. Then, in view of lemma 1A, we immediately have that $\hat{G}_0[y]$ is asymptotically normally distributed with mean $G_0[y]$ and variance

$$v_{\hat{G}_0}[y] = \frac{1}{n}\left(\left(\frac{1}{\mu_1[y]}\right)^2 \psi_1[y] - 2\left(\frac{1}{\mu_1[y]}\right)\psi_{1r}[y] + \psi_r[y]\right)$$

It is possible to show

Theorem A2: Under the null hypothesis of equality of $\hat{G}_0[y]$ and $\hat{G}_0[y']$, $\hat{G}_0[y] - \hat{G}_0[y']$ is asymptotically normally distributed with mean zero and asymptotic variance given by

$$v_{\hat{G}_0[y]-\hat{G}_0[y']} = \frac{1}{n} g_0[y,y']' \Psi_{1r}[y,y'] g_0[y,y']$$

where $g_0[y,y'] = \begin{bmatrix} \frac{1}{\mu_1[y]} & -1 & -\frac{1}{\mu_1[y']} & 1 \end{bmatrix}$

ii) When θ=1 the generalised entropy index reduces to the Theil index and equals

$$G_1[y] = \frac{\lambda[y]}{\mu_1[y]} - \log[\mu_1[y]]$$

where $\lambda[y] = \int_0^\infty y \log[y] dF[y]$. The natural estimator of $\lambda[y]$ is $l[y] = \frac{1}{n}\sum_{i=1}^{n} y_i \log[y_i]$, so that the natural estimator for $G_1[y]$ is

$$\hat{G}_1[y] = \frac{l[y]}{m_1[y]} - \log[m_1[y]]$$

We can now formulate a lemma, the proof of which is similar to the proof of the lemma in the main text:

Lemma A2:

$\left[\sqrt{n}(m_\alpha[y] - \mu_\alpha[y]) \quad \sqrt{n}(l[y] - \lambda[y]) \quad \sqrt{n}(m_\alpha[y'] - \mu_\alpha[y']) \quad \sqrt{n}(l[y'] - \lambda[y'])\right]$ is asymptotically multivariate normally distributed with mean zero and variance-covariance matrix

$$\Xi_{\alpha\lambda}[y,y'] = \begin{bmatrix} \xi_{\alpha}[y] & \xi_{\lambda\alpha}[y] & \xi_{\alpha\alpha}[y',y] & \xi_{\lambda\alpha}[y',y] \\ \xi_{\alpha\lambda}[y] & \xi_{\lambda}[y] & \xi_{\alpha\lambda}[y',y] & \xi_{\lambda\lambda}[y',y] \\ \xi_{\alpha\alpha}[y,y'] & \xi_{\lambda\alpha}[y,y'] & \xi_{\alpha}[y'] & \xi_{\lambda\alpha}[y'] \\ \xi_{\alpha\lambda}[y,y'] & \xi_{\lambda\lambda}[y,y'] & \xi_{\alpha\lambda}[y'] & \xi_{\lambda}[y'] \end{bmatrix}$$

where

$$\xi_{\alpha}[y] = \omega_{\alpha\alpha}[y,y]$$

$$\xi_{\alpha\alpha}[y,y'] = \omega_{\alpha\alpha}[y,y']$$

$$\xi_{\lambda}[y] = E\{(y\log[y])^2\} - (\lambda[y])^2$$

$$\xi_{\lambda\lambda}[y,y'] = E\{y\log[y]y'\log[y']\} - \lambda[y]\lambda[y']$$

$$\xi_{\alpha\lambda}[y,y'] = E\{y^{\alpha}y'\log[y']\} - \mu_{\alpha}[y]\lambda[y']$$

$$\xi_{\alpha\lambda}[y] = E\{y^{\alpha+1}\log[y]\} - \mu_{\alpha}[y]\lambda[y] = \xi_{\lambda\alpha}[y]$$

$$\xi_{\lambda\alpha}[y,y'] = E\{(y')^{\alpha}y\log[y]\} - \mu_{\alpha}[y']\lambda[y]$$

Each of the elements of $\Xi_{\alpha\lambda}[y,y']$ can be consistently estimated in the usual way. Because of Rao's theorem, $\hat{G}_1[y]$ is asymptotically normally distributed with mean $G_1[y]$ and variance

$$v_{\hat{G}_1}[y] = \frac{1}{n}\left(\left(\frac{\lambda[y]}{(\mu_1[y])^2} + \frac{1}{\mu_1[y]}\right)^2 \xi_1[y] - 2\left(\frac{\lambda[y]}{(\mu_1[y])^2} + \frac{1}{\mu_1[y]}\right)\left(\frac{1}{\mu_1[y]}\right)\xi_{1\lambda}[y] + \left(\frac{1}{\mu_1[y]}\right)^2 \xi_{\lambda}[y]\right)$$

In addition, we can establish theorem A3:

Theorem A3: Under the null hypothesis of equality of $\hat{G}_1[y]$ and $\hat{G}_1[y']$, $\hat{G}_1[y] - \hat{G}_1[y']$ is asymptotically normally distributed with mean zero and asymptotic variance given by

$$v_{\hat{G}_1[y]-\hat{G}_1[y']} = \frac{1}{n} g_1[y,y']' \,\Xi_{1\lambda}[y,y'] g_1[y,y']$$

where $g_1[y,y'] = \begin{bmatrix} -\frac{\lambda[y]}{(\mu_1[y])^2} - \frac{1}{\mu_1[y]} & \frac{1}{\mu_1[y]} & \frac{\lambda[y']}{(\mu_1[y'])^2} + \frac{1}{\mu_1[y']} & -\frac{1}{\mu_1[y']} \end{bmatrix}$

REFERENCES

Cowell, F. A., 1989, Sampling variance and decomposable inequality measures, Journal of Econometrics 42, 27-41.

Cowell, F.A., 1995, Measuring inequality (Prentice Hall/ Harvester Wheatsheaf, Hemel Hampstead).

Cowell, F.A. and M.-P. Victoria-Feser, 1996, Robustness properties of inequality measures, Econometrica 64, 77-101.

Davidson, R. and J.-Y. Duclos, 1997, Statistical incidence for the measurement of the incidence of taxes and transfers, Econometrica 65, 1453-1465.

Duclos, J.-Y., 1997, The asymptotic distribution of linear indices of inequality, progressivity and redistribution, Economics Letters 54, 51-57.

Kiefer, D. W., 1985, Distributional tax progressivity indices, National Tax Journal 37, 497-513.

Lehmann, E.L., 1983, Theory of point estimation (John Wiley & Sons, New York).

Mills, J. and S. Zandvakili, 1997, Statistical inference via bootstrapping for measures of inequality, Journal of Applied Econometrics 12, 133-150.

Thistle, P. D., 1990, Large sample properties of two inequality indices, Econometrica 58, 725-728.