Maynooth University
National University of Ireland Maynooth

Department of Economics Finance & Accounting

Working Paper N294-19

Semi-strong factors in asset returns*

Gregory Connor[†] Robert A. Korajczyk [‡]

July 13, 2019

*We would like to thank Juan Carlos Alismendri-Zambrano, Thomas Flavin, Oliver Linton, Donal O'Neill, Anita Suurlaht and seminar participants at Cambridge University and University College Dublin for helpful comments. We thank Honghao Wang for excellent research assistance. Connor acknowledges the support of Science Foundation Ireland under Grant Number 16/SPP/3347.

[†]Department of Economics, Finance and Accounting, Maynooth University, Maynooth, Ireland. E-mail: gregory.connor@mu.ie.

[‡]Department of Finance, Kellogg School of Management, Northwestern University, Evanston, IL, USA. E-mail: r-korajczyk@kellogg.northwestern.edu.

Abstract

This paper refines the approximate factor model of asset returns by dividing systematic factors into a) natural rate factors, whose sum of squared factor betas grow at the same rate as the number of assets, and b) semi-strong factors, whose sum of squared factor betas grow, but at a slower rate. We describe a methodology to estimate the cross-sectional mean and mean-square of semi-strong factor betas, and to differentiate them from natural rate factors. We apply the methodology to US equity returns using daily changes in exchange rates and commodity prices as semi-strong factors. We find that oil and gold price changes are significant factors while foreign exchange rate changes are only significant in more recent subperiods.

1 Introduction

The approximate factor model of asset returns is a workhorse of research on asset pricing and portfolio construction. It uses the many-asset framework of Ross's (1976) Arbitrage Pricing Theory, in which we create approximate relationships for a many-asset economy by imagining a sequence of economies with a growing number of assets, and rely on limiting conditions as the number of assets goes to infinity. In the classic formulation of Chamberlain and Rothschild (1983), an approximate factor model is defined by the asymptotic behavior of the eigenvalues of the asset return covariance matrix when the number of assets, n, approaches infinity. If the largest k eigenvalues of the covariance matrix grow unboundedly with n and the remaining eigenvalues are bounded above for all n, then returns obey an approximate factor model with k factors. Chamberlain and Rothschild also show that the eigenvectors of the return covariance matrix associated with the k largest eigenvalues can serve as factor exposures (factor betas) up to an arbitrary nonsingular rotation of the factors.

Chamberlain and Rothschild work with the true covariance matrix and do not directly address estimation issues. In empirical implementations of their framework, it is standard to impose slightly more structure. In particular, standard empirical implementations assume that the k dominant eigenvalues grow at the "natural" rate of n. This paper considers the case in which the Chamberlain and Rothschild assumptions hold, but some eigenvalues grow at a slower rate, $n^{1-2\alpha}$, $0 < \alpha < \frac{1}{2}$. We show that the associated factors can be interpreted as common, but less important, sources of return variation. Following the terminology of Chudik, Pesaran and Tosetti (2011), we call these *semi-strong* factors. Semi-strong factors are stronger than *weak factors* which have bounded eigenvalues, but weaker than *natural rate* factors whose eigenvalues grow at the rate n.[1]

We work with the case in which factors are observed directly, so the eigenvalue condition is more simply stated as a condition on the sum of squared betas for a given factor. A natural rate factor has sum of squared betas going to infinity at rate n whereas a semi-strong factor has sum of squared betas going to infinity at rate $n^{1-2\alpha}$. We estimate the factor betas with simple time series ordinary least squares of individual returns data regressed on the set of factors. We show that by using the large cross-section of time-series regression-estimated factor betas one can measure cross-sectional features such as the average factor beta and the mean-squared factor beta. We derive the asymptotic distributions of the cross-sectional mean and mean-square of the time-series estimated factor betas, and propose test statistics for distinguishing them from zero and for distinguishing natural rate from semi-strong factors.

As a concrete example of an application of our framework, consider the sensitivity of individual US equities to fluctuations in the South Korean Won exchange rate against an international currency basket. Some US companies (e.g., Korean goods importers) may benefit favorably from a relative decline in the Korean Won exchange rate, and some others (e.g., domestic manufacturers competing against Korean imports) may benefit from a relative increase. The vast majority of firms will have fairly negligible betas, and it is difficult to justify including the Korean Won exchange rate as one of the small number of important

[1] Note that the term "grow" is used here in terms of the asymptotic approximation for large n; in actual data, n is fixed. A difference in "rate of growth" means in practice that the approximation for natural rate factors has different statistical properties than the approximation for semi-strong factors.

factors in a model of US equity returns. If an investor holds any well-diversified portfolio of equities, the portfolio exposure to Korean Won relative exchange rate fluctuations will be negligible. Economists may still be interested in measuring the cross-sectional features of assets' exposures to this small common factor, even it is only a negligible source of risk in well-diversified portfolios. For example, we can examine whether the aggregate features of the cross-section of individual betas of US firms to Korean Won movements has changed over time. We may wish to measure the magnitude of the (fairly small) influence of Korean exchange rate changes on US equities, and examine whether the changing nature of the trade relationship between the US and Korea is reflected in the changing magnitude of this influence over subperiods. Portfolio analysts may be interested in measuring or hedging Korean Won risk if for some reason they are holding a poorly diversified portfolio which included a non-negligible position in this risk.

Section 2 presents the statistical framework for our analysis. We use a random coefficients model to generate both natural rate and semi-strong factors within a unified approximate factor model framework. Section 3 presents theoretical results on the estimation and testing of factor betas and their cross-sectional distribution (average and mean-square). We give conditions such that the factor betas' cross-sectional moments can be consistently estimated from a large-n cross-section of individual fixed-T time-series regression coefficient estimates. We also devise a statistical test for distinguishing the two types of factors.

Section 4 applies the methodology to individual US equity returns using the Fama-French market, size and value indices and changes in foreign exchange rates and commodity prices as factors. We use daily data over six five-year subperiods, 1989-2018. We find that the Fama-French indices are natural rate factors. Oil and gold price changes are statistically significant factors, but the evidence is mixed as to whether they are better classified as natural rate or semi-strong. Foreign exchange factors are significant only in the more recent subperiods and are only semi-strong factors. Section 5 concludes.

2 An approximate factor model with semi-strong factors

We begin by imposing a factor model on returns, and then differentiate between natural-rate and semi-strong factors within the model. We consider a sequence of return models with increasing cross-sectional dimension. For each n, let r_t denote the $n-$vector of asset returns. We assume that for each n the vector of returns obeys the following model:

$$r_t = a + Bf_t + \varepsilon_t,\ \ t = 1, T, \tag{1}$$

where a is an $n-$vector of constants, B is a $n \times k-$matrix of factor betas, f_t is a $k-$vector of random factors, and ε_t is an $n-$vector of idiosyncratic returns. We assume that f_t and ε_t are mutually independent and each is distributed i.i.d. through time. All elements of f_t and ε_t are assumed to have zero means without loss of generality, due to the inclusion of the vector of constants, a. Let $\Omega_r = E[(r-a)(r-a)']$, $\Omega_f = E[ff']$, and $\Omega_\varepsilon = E[\varepsilon\varepsilon']$ denote the covariance matrices; we assume that all three are nonsingular. All the elements in (1) implicitly include an n superscript, since we are working with asymptotic properties not fixed-n properties, but we leave this implicit throughout. Of course, when we apply the econometric model to actual data, the number of assets n will be a constant; the increasing$-n$ framework is just an asymptotic model to derive meaningful approximations for large n. The time-series sample size T is fixed throughout the analysis.

We impose the standard Chamberlain-Rothschild assumptions that the largest k eigenvalues of the return covariance matrix increase unboundedly with n and the remaining eigenvalues are bounded. Let $||\cdot||$ denote the Euclidian matrix norm over square matrices, $||A|| = \sup_x \frac{|x'Ax|}{\sqrt{(x'x)}}$; recall that in the case of a symmetric positive semidefinite matrix this equals the largest eigenvalue. Since ε_t captures the idiosyncratic risks, we impose:

$$||\Omega_\varepsilon|| \leq c_1 \text{ for all } n. \tag{2}$$

We also impose the Chamberlain-Rothschild unboundedness assumption on the first k eigenvalues, not restricting their rate of increase. Note that $\Omega_r = B\Omega_f B' + \Omega_\varepsilon$. Since Ω_ε has bounded eigenvalues by assumption above, and the eigenvalues of Ω_f are fixed, k unbounded eigenvalues for Ω_r is identical to assuming that all k eigenvalues of the inner product matrix of the betas, $(B'B)$, increase without bound:

$$\lim_{n\to\infty} (\min eigval[B'B]) = \infty. \tag{3}$$

Note that (3) implies that all the eigenvalues of $B'B$ are strictly positive except possibly for $n < \overline{n}$ for some finite $\overline{n}$. All eigenvalues strictly positive means that $B'B$ is nonsingular.

Therefore, except possibly on a finite subset of $n < \overline{n}$, (3) can be written equivalently as:

$$\lim_{n\to\infty} ||(B'B)^{-1}|| = 0, \tag{4}$$

since the eigenvalues of a symmetric positive definite matrix are the reciprocals of the eigenvalues of its inverse.

We keep the two assumptions (2) and (3) throughout. We note however that these two assumptions alone are not sufficient for empirical work. Empirical implementations typically strengthen (3) slightly, in particular by requiring that all k dominant eigenvalues increase at the "natural" rate n. That is, most empirical analysts replace (3) with:

$$\lim_{n\to\infty} \frac{1}{n} \min eigval[B'B] > 0, \tag{5}$$

or equivalently (ignoring some technically uninteresting cases[2]):

$$\lim_{n\to\infty} \frac{1}{n}(B'B) = M, \text{ a nonsingular matrix.} \tag{6}$$

which differs from (4) in the speed at which the eigenvalues must grow with n; (6) requires that the eigenvalues of $B'B$ go to infinity at rate n whereas (4) only requires that they go to infinity, at any rate.

We call (6) the "natural" rate of increase since many straightforward rules for choosing the series of factor beta matrices will result in (6). For example, for notational simplicity let $k = 1$ and let $B = \{+1, -1, +1, -1, ...\}$ where the simple pattern $+1, -1$ is repeated $n/2$

[2] We must exclude the case in which the beta matrix grows explosively with n, and also exclude cases in which the $k \times k-$matrix series $\frac{1}{n}B'B$ only converges on a subsequence or on multiple subsequences. These cases are automatically excluded with probability one in our random coefficients framework described later.

times. This, or any other repeated pattern of a nonzero set of numbers, will give (6) for $k = 1$. This extends simply to $k > 1$ replacing a nonzero pattern with a nonsingular matrix pattern (as long as the generating matrix pattern has rank k). Condition (6) will also hold with probability one if the coefficients in B are chosen randomly (over $i = 1, n$) from an i.i.d. process with nonsingular covariance matrix, by the strong law of large numbers (shown later).

By using a less straightforward rule for choosing the sequence of beta matrices, the natural rate can be replaced with a slower, semi-strong, rate of increase; this is easily done by scaling fixed factor betas so that the individual betas shrink with n. So, as a simple example with $k = 1$, consider the sequence of $n-$vectors $B^n = \{-\frac{1}{n^{1/3}}, +\frac{1}{n^{1/3}}, -\frac{1}{n^{1/3}}, +\frac{1}{n^{1/3}}, ...\}$. This fulfils the Chamberlain-Rothschild condition (4), since $B^{n\prime}B^n = (n \cdot n^{-2/3}) = n^{1/3} \to \infty$, but not the natural rate condition (6), since $\frac{1}{n}B^{n\prime}B^n = (n^{-1} \cdot n \cdot n^{-2/3}) = n^{-2/3} \to 0$. Stated simplistically and intuitively, semi-strong factors have sum of squared factor betas going to infinity but average squared factor beta going to zero with n. A more precise definition appears later.

Our empirical implementation does not require the natural rate assumption; it allows some of dominant k eigenvalues to have a slower, semi-strong rate of increase. To give structure to the problem, we generate the factor exposure matrix B with a random coefficients model as described in the next subsection.

2.1 Generating factor betas with a random coefficients model

In our factor model, the beta matrix and factors divide into two subsets with k^{nr} and k^{ss} elements, depending upon the average cross-sectional magnitude of the factor betas:

$$Bf = B^{nr}f^{nr} + B^{ss}f^{ss} \tag{7}$$

where the superscript nr denotes "natural rate" and ss denotes "semi-strong." The beta matrix B will be generated from the first n random draws of an i.i.d. sequence of realizations of a $k-$vector random variable b with finite mean vector and nonsingular covariance matrix:

$$E[b] = \mu_b;\ E[(b-\mu_b)(b-\mu_b)'] = \Omega_b;$$

note that the block-diagonal submatrices Ω_b^{nr} and Ω_b^{ss} are also nonsingular given that Ω_b is nonsingular. We also assume that all k components of the random vector b have finite fourth moments. We use the sequence of random draws of b to create the factor beta matrix, scaling appropriately to differentiate between natural-rate and semi-strong factors.

Let B^* denote the $n \times k-$matrix of the first n draws of the vector stochastic process b. We call B^* the beta generating matrix. Partition the beta generating matrix into the first k^{nr} and remaining k^{ss} columns; $B^* = [B^{nr*}|B^{ss*}]$. The first k^{nr} factors are natural rate factors; their betas are simply the random draws of the first k^{nr} elements of $b:$

$$B^{nr} = B^{nr*};$$

The remaining $k^{ss} = k - k^{nr}$ factors are semi-strong. We fix a rate exponent $\alpha > 0$ and assume that the factors betas are of the order $n^{-\alpha}$; they are generated by scaled versions of the generating matrix:

$$B^{ss} = (n^{-\alpha})B^{ss*}. \tag{8}$$

We impose the condition $\alpha < \frac{1}{2}$. As we will discuss later, this condition is required in order for the sum of squared factor betas to increase with n. If we were to allow $\alpha > \frac{1}{2}$ we would generate "semi-weak" factors, using the terminology of Chudik et al. (2011).

It is possible (although cumbersome) to allow α to vary across the semi-strong factors; we do not pursue that here. Note also that the natural rate factors could be said to obey (8) by allowing $\alpha = 0$ for them, since $n^{-0} = 1$ for all n.

Since the vector stochastic process b is i.i.d. and has finite fourth moments, it follows from the strong law of large numbers (see White (1984, Proposition 2.11)) that the first two cross-sectional moments of B converge almost surely, so that:

$$\lim_{n\to\infty} (\frac{1}{n})B^{nr\prime}B^{nr} = \Omega_b^{nr}, \text{ a.s.}, \tag{9}$$

and:

$$\lim_{n\to\infty} (\frac{1}{n^{1-2\alpha}})B^{ss\prime}B^{ss} = \lim_{n\to\infty} (\frac{n^{2\alpha}}{n^{2\alpha}})(\frac{1}{n})B^{ss*\prime}B^{ss*} = \Omega_b^{ss}, \text{ a.s.}, \tag{10}$$

and the cross-product matrix:

$$\lim_{n\to\infty} (\frac{1}{n^{1-\alpha}})B^{ss\prime}B^{nr} = \lim_{n\to\infty} (\frac{n^{\alpha}}{n^{\alpha}})(\frac{1}{n})B^{ss*\prime}B^{nr*} = \Omega_b^{ss,nr}, \text{ a.s..} \tag{11}$$

The strong law of large numbers also applies to the first moment vectors (appropriately scaled):

$$\lim_{n\to\infty}(\frac{1}{n})B^{nr\prime}1^n = \mu_b^{nr}, \text{ a.s.,} \tag{12}$$

and:

$$\lim_{n\to\infty}(\frac{1}{n^{1-\alpha}})B^{ss\prime}1^n = \lim_{n\to\infty}(\frac{n^\alpha}{n^\alpha})(\frac{1}{n})B^{ss*\prime}1^n = \mu_b^{ss}, \text{a.s..} \tag{13}$$

From here on in the paper, we condition on a particular sequence of random realizations of B and so treat B for estimation purposes as a nonrandom matrix. We assume that the particular matrix sequence obeys the almost-sure limit conditions (9),(10), (11), (12) and (13).

3 Estimation of betas and their cross-sectional average and mean-square

We assume that the factors are observed exactly, so that the statistical problem involves estimating the factor betas, and features of their cross-sectional distribution (in particular, their average and mean-square). Since f and ε are assumed mutually independent with ε conditionally mean zero, (1) gives a seemingly unrelated regression model without cross-equation restrictions:

$$r_{it} = a_i + B_i f_t + \varepsilon_{it};\ i = 1, n; t = 1, T. \tag{14}$$

We condition upon f_t $t = 1, T$ and so treat it for the purposes of parameter estimation as a nonrandom time series of fixed length T. Since there are no cross-equation restrictions, the regression system (14) decomposes into n separate time-series ordinary least squares problems. Let F denote the $T \times (k+1)-$matrix of the explanatory variables in (14), where the first column is a $T-$vector of ones, and column $j+1$ is the time-series of factor realizations for factor j; the regression estimates for any particular i are:

$$\{\widehat{a}_i, \widehat{B}_i\} = (F'F)^{-1}F'R_i,$$

where R_i is the $T-$vector of asset returns. By standard regression theory for any j we can describe $\widehat{B}_{ij}$ as the true value plus a linear combination of the residuals, in particular:

$$\widehat{B}_{ij} = B_{ij} + \sum_{t=1}^{T} m_{jt}\varepsilon_{it}, \tag{15}$$

where $m_{jt} = [(F'F)^{-1}F']_{j+1,t}$. The ordinary least squares estimates are unbiased, with estimation variance:

$$E[(\widehat{B}_{ij} - B_{ij})^2] = [(F'F)^{-1}]_{j+1,j+1}\sigma^2_{\varepsilon_i}. \tag{16}$$

We consider the estimation-error corrected squared regression coefficient:

$$(\widehat{B}_{ij})^2 - [(F'F)^{-1}]_{j+1,j+1}\widehat{\sigma}^2_{\varepsilon_i}, \tag{17}$$

where $\widehat{\sigma}^2_{\varepsilon_i} = \frac{1}{T-k-1}\sum_{t=1}^{T}\widehat{\varepsilon}^2_{it}$, which is an unbiased estimate of $\sigma^2_{\varepsilon_i}$ by standard regression theory. The estimate $\widehat{\sigma}^2_{\varepsilon_i}$ can be written as a quadratic function of ε_{it}, $t = 1, T$, in particular:

$$\widehat{\sigma}^2_{\varepsilon_i} = \sum_{t=1}^{T}\sum_{\tau=1}^{T} p_{t\tau}\varepsilon_{it}\varepsilon_{i\tau}.$$

where $p_{t\tau} = (\frac{1}{T-k-1})[I - F(F'F)^{-1}F']_{t,\tau}$.

The estimation-error-adjusted squared beta (17) is a key input to our mean-squared beta estimate in the next subsection. Since its two components, $\widehat{B}_{ij}$ and $\widehat{\sigma}^2_{\varepsilon_i}$, are (respectively) linear and quadratic functions of the innovations ε_{it}, $t = 1, T$, its estimation variance is a linear combination of the first four moments and squared second moment of ε_i. The weights in this linear combination are products and cross-products of m_{jt} and $p_{t\tau}$, $t, \tau = 1, T$.

Define $\overline{m}_{2j}$, $\overline{m}_{3j}, \overline{m}_{4j}$ as the second, third and fourth power sums of m_{jt}, $t = 1, T$, and $\overline{m}_{22j}$ as the sum of cross-products:

$$\begin{aligned}
\overline{m}_{2j} &= \sum_{t=1}^{T} m_{jt}^2 \\
\overline{m}_{3j} &= \sum_{t=1}^{T} m_{jt}^3 \\
\overline{m}_{4j} &= \sum_{t=1}^{T} m_{jt}^4 \\
\overline{m}_{22j} &= \sum_{t=1}^{T} \sum_{\tau \neq t} m_{j\tau}^2 m_{jt}^2.
\end{aligned}$$

Define the following product sums and cross-product sums of $p_{t\tau}$, $t = 1, T; \tau = 1, T$:

$$\begin{aligned}
\overline{p}_2 &= \sum_{t=1}^{T} p_{tt}^2 \\
p \times p &= \sum_{t=1}^{T} \sum_{\tau \neq t} (2p_{t\tau}^2 + p_{tt} p_{\tau\tau}) \\
\overline{pm} &= \sum_{t=1}^{T} p_{tt} m_{jt} \\
\overline{pm}_2 &= \sum_{t=1}^{T} p_{tt} m_{jt}^2 \\
p \times m &= \sum_{t=1}^{T} \sum_{\tau \neq t} (2p_{t\tau} m_{jt} + p_{tt} m_{jt}^2)
\end{aligned}$$

By straightforward algebra (see Appendix 1 for detailed steps) the estimation variance of $\widehat{B}_{ij}^2 - [(F'F)^{-1}]_{j+1,j+1} \widehat{\sigma}_{\varepsilon_i}^2$ is:

$$Var[(\widehat{B}_{ij})^2 - \overline{m}_{2j} \widehat{\sigma}_{\varepsilon_i}^2] = c_{1ij} \sigma_{\varepsilon_i}^2 + c_{2ij} E[\varepsilon_i^3] + c_{3j} E[\varepsilon_i^4] + c_{4j} (\sigma_{\varepsilon_i}^2)^2, \tag{18}$$

where:

$$
\begin{aligned}
c_{1ij} &= 4\overline{m}_{2j}B_{ij}^2 \\
c_{2ij} &= 4B_{ij}\overline{m}_{3j} - 4\overline{m}_{2j}B_{ij}(\overline{pm}_j) \\
c_{3j} &= \overline{m}_{4j} + (\overline{m}_{2j})^2\overline{p}_2 - 2\overline{m}_{2j}(\overline{pm}_{2j}) \\
c_{4j} &= 3\overline{m}_{22j} + (\overline{m}_{2j})^2(p \times p) - 2\overline{m}_{2j}(p \times m)
\end{aligned}
$$

Readers will be pleased to know (or at least relieved) that we do not estimate this complicated variance expression directly, but instead invoke bootstrap methods (see below).

3.1 Statistical identification of the cross-sectional average and mean-square of semi-strong factor betas

In the case of a semi-strong factor, estimates of the individual betas have limited information since $B_{ij} = n^{-\alpha}B_{ij}^*$ implies that the magnitude of most betas is too small to detect reliably for large n. However, as we now show, it is possible to detect the aggregate influence of semi-strong factors by averaging the cross-section of individual time-series regression estimates (and their squares). Define μ_j and χ_j, $j = 1, k$, as the cross-sectional averages and average sums of squares of the factor betas:

$$
\begin{aligned}
\mu_j &= \frac{1}{n}\sum_{i=1}^{n} B_{ij}; \\
\chi_j &= \frac{1}{n}\sum_{i=1}^{n} B_{ij}^2; \;\; j = 1, k,
\end{aligned}
$$

We use the cross-section of estimated betas to construct simple estimation-error-adjusted estimates:

$$\widehat{\mu}_j = \frac{1}{n}\sum_{i=1}^{n}\widehat{B}_{ij}; \tag{19}$$

$$\widehat{\chi}_j = \frac{1}{n}\sum_{i=1}^{n}[\widehat{B}_{ij}^2 - \overline{m}_{2j}\widehat{\sigma}_{\varepsilon_i}^2]. \tag{20}$$

We assume that ε_i, $i = 1, n$ is independently distributed across i, with bounded fourth moments for all i. It is possible in theory to extend our results to the case in which the idiosyncratic returns are weakly cross-sectionally correlated, but that would substantially complicate the derivation of the estimation variance and we do not pursue that extension here. Let m denote any fixed, finite $T-$vector (constant across $i = 1, n$) and let $\widetilde{\varepsilon}_i$ denote a $T-$vector of independent realization of ε_i. For a given $j \in [1, 2, ...k]$ define the random series $\gamma_i = m'\widetilde{\varepsilon}_i$ and $\theta_i = 2B_{ij}m'\widetilde{\varepsilon}_i + (m'\widetilde{\varepsilon}_i)^2$. We assume that for any fixed m the Lindeberg condition (see White (1984, pp 111)) holds for both γ_i and θ_i. Since the estimation error in $\widehat{B}_{ij}$ has the form $m'\widetilde{\varepsilon}_i$, this guarantees that the central limit theorem applies to cross-sectional averages of beta estimates and squared beta estimates. Since the regression estimates are independent across i, the variance of the cross-sectional average is $\frac{1}{n}$ times the average of the variances. Using (16) and (18):

$$\begin{aligned} nVar[\widehat{\mu}_j] &= \frac{1}{n}\sum_{i=1}^{n}\overline{m}_{2j}\sigma_{\varepsilon_i}^2 \\ nVar[\widehat{\chi}_j] &= \frac{1}{n}\sum_{i=1}^{n}c_{1ij}\sigma_{\varepsilon_i}^2 + c_{2ij}E[\varepsilon_i^3] + c_{3j}E[\varepsilon_i^4] + c_{4j}(\sigma_{\varepsilon i}^2)^2. \end{aligned} \tag{21}$$

We assume that the cross-sectional averages converge for large n and let ϕ and ψ_j denote these convergent limits:

$$\phi = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\sigma_{\varepsilon_i}^2 \tag{22}$$

$$\psi_j = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} c_{1ij}\sigma_{\varepsilon_i}^2 + c_{2ij}E[\varepsilon_i^3] + c_{3j}E[\varepsilon_i^4] + c_{4j}(\sigma_{\varepsilon i}^2)^2. \tag{23}$$

It is straightforward to derive the convergence conditions (22) and (23) from more fundamental assumptions, for example by placing bounds on the first four moments of ε_i for all i, and restricting the co-relations between B_{ij} and these moments for all i. This additional complexity would add little value to the analysis so we simply directly assume that these limits exist.

Since the test statistics are asymptotically normal we have:

$$d\lim_{n\to\infty} n^{\frac{1}{2}}(\widehat{\mu}_j - \mu_j) \sim N(0, \overline{m}_{2j}\phi) \tag{24}$$

and:

$$d\lim_{n\to\infty} n^{\frac{1}{2}}(\widehat{\chi}_j - \chi_j) \sim N(0, \psi_j),\ \ j = 1, k. \tag{25}$$

Note that for semi-strong factors $j > k^{nr}$ we have $\lim_{n\to\infty} \mu_j = \lim_{n\to\infty} n^{-\alpha}\mu_j^* = 0$ and $\lim_{n\to\infty} \chi_j = \lim_{n\to\infty} n^{-2\alpha}\chi_j^* = 0$. As long as $\alpha < \frac{1}{2}$ then the test statistic (24) has rejection power against H0: $\mu_j^* = 0$ approaching one for $n \to \infty$ whenever the true mean is different from zero. Test statistic (25) requires the stronger condition $\alpha < \frac{1}{4}$ for unit asymptotic power against H0: $\chi_j^* = 0$.

The population value of mean-squared beta is always positive if the true mean beta is nonzero, whereas it is possible to have a positive mean-squared beta when the true mean beta is exactly zero. This might give the false impression that the mean-squared beta test "dominates" the mean beta test, but the relationship between the statistical tests for nonzero mean-squared beta and nonzero mean beta is actually ambiguous. The estimation error in the test of mean-square beta can be much higher than in the test of mean beta, so either test can reject when the other does not. In the case of semi-strong factors, in terms of asymptotic power, the test for a nonzero mean beta only requires $\alpha < \frac{1}{2}$ whereas the test for positive mean-squared beta requires $\alpha < \frac{1}{4}$. In this sense, the mean beta test is asymptotically stronger than the mean-squared beta test.

3.2 Bootstrap estimation of the variances

We have derived an explicit formula for the variances of the mean-squared beta estimates, $\psi_j/n \; j = 1, k$, in (21), but the formula involves the cross-sectional average of a complicated linear combination of higher moments of asset-specific returns. A naive approach would be to proxy each term in (21) with time-series estimates from each asset in the sample, and then take cross-sectional averages. Instead we use a bootstrap methodology, which obviates the need to estimate the individual components of (21); we discuss our motivation and justification for this bootstrapped estimator in more detail in Appendix 2. The procedure is as follows. Let F denote the $k \times T$ matrix of factor realizations. First, we randomly resample with replacement the $T-$vector of dates $t = 1, T$ producing a random mapping $t^*(t); t = 1, T$. We then replace the matrix of returns R and factors F with the resampled alternatives R^* and F^* based on this random selection of dates:

$$\begin{aligned} R_{it}^* &= R_{it^*(t)};\ i = 1, n; t = 1, T \\ F_{jt}^* &= F_{jt^*(t)};\ j = 1, k; t = 1, T. \end{aligned}$$

Using R^* and F^* in place of R and F we re-estimate $\widehat{B}$ and $\widehat{\chi}_j$, and repeat this random bootstrap a large number of times, $h = 1, m$. For each j, the variances $n\widehat{\psi}_j$ are found from the collection of m bootstrapped estimates of $\widehat{\chi}_j$:

$$n\widehat{\psi}_j = \frac{1}{m}\sum_{h=1}^{m}(\widehat{\chi}_j^h)^2 - (\frac{1}{m}\sum_{h=1}^{m}\widehat{\chi}_j^h)^2.$$

3.3 Distinguishing semi-strong from natural-rate factors

We now consider the problem of empirically distinguishing semi-strong factors from natural rate factors. Either type of factor has sum of squared betas going to infinity with n, but a natural rate factor has mean-squared beta converging to a strictly positive value whereas a semi-strong factor has mean-squared beta going to zero. For large n, the semi-strong factors are statistically significant (in terms of their mean squared betas being testably greater than zero as the estimation variance shrinks) but also inconsequential (in terms of their contribution to average explanatory power going to zero). In the subsections above we have developed an estimation method which allows us to identify both natural-rate and semi-strong factors. Now in this subsection we attempt to find methods to distinguish between the two types. First we propose the use of marginal R-square of each factor in our model as a descriptive statistic; semi-strong factors will have "small" values for marginal R-square. Then we present a formal test statistic using a penalized version of our adjusted mean-squared beta estimate.

3.3.1 Marginal explanatory power of each factor

The seemingly unrelated regression system (14) can be viewed as a single stacked regression model with n intercepts (also called asset fixed effects) and nk factor betas, with heteroskedasticity across the n assets. Define the adjusted R-squared for this stacked regression model as one minus unexplained variance over total variance:

$$\overline{R}^2 = 1 - (\frac{(T-1)}{(T-k-1)})(\frac{1}{nT}\sum_{t=1}^{T}\sum_{i=1}^{n}\widehat{\varepsilon}_{it}^2)/(\frac{1}{nT}\sum_{t=1}^{T}\sum_{i=1}^{n}(r_{it}-\overline{r}_i)^2),$$

where $\overline{r}_i$ is the time-series sample mean return of asset i. Next, we re-estimate the linear regression model with all factors except factor j, and re-calculate the stacked regression $\overline{R}^2$ using the newly estimated residuals $\widehat{\varepsilon}_{it}^{*2}$:

$$\overline{R}_{(j)}^2 = 1 - (\frac{(T-1)}{(T-k)})(\frac{1}{nT}\sum_{t=1}^{T}\sum_{i=1}^{n}\widehat{\varepsilon}_{it}^{*2})/(\frac{1}{nT}\sum_{t=1}^{T}\sum_{i=1}^{n}(r_{it}-\overline{r}_i)^2).$$

The difference between $\overline{R}^2$ and $\overline{R}_{(j)}^2$ gives our marginal R^2 statistic.

$$\Delta\overline{R}_j^2 = \overline{R}^2 - \overline{R}_{(j)}^2. \tag{26}$$

The intuition behind (26) is that if factor j is only semi-strong, then the marginal R-squared should be small for large n, since $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(B_{ij})^2 = 0$. To understand the intuition, define $\sigma_{fj}^2 = \frac{1}{T}\sum_{t=1}^{T}f_{jt}^2$ and $\sigma_r^2 = \frac{1}{nT}\sum_{t=1}^{T}\sum_{i=1}^{n}(r_{it}-\overline{r}_i)^2$. Consider the population equivalents of the two R-squareds, that is, $R^2 = 1-(\frac{1}{n}\sum_{i=1}^{n}\sigma_{\varepsilon i}^2)/\sigma_r^2$ and $R_{(j)}^2 = 1-(\frac{1}{n}\sum_{i=1}^{n}\sigma_{\varepsilon i}^2 + B_{ij}^2\sigma_{fj}^2)/\sigma_r^2$. Taking the difference between these two population R-squareds gives:

$$\Delta R_j^2 = \frac{\sigma_{fj}^2}{\sigma_r^2}\left(\frac{1}{n}\sum_{i=1}^{n} B_{ij}^2\right), \tag{27}$$

hence the population marginal R-squared is a scaled version of mean-squared beta. The estimated marginal R-squared (26) is not a formal test statistic since we do not give its sample distribution, but it is a useful descriptive statistic based on (27).

3.3.2 A penalized test statistic for mean-squared beta

We now construct a test statistic for empirically distinguishing natural-rate factors from semi-strong factors. Either type of factor has sum of squared betas going to infinity with n, but a natural rate factor has mean-squared beta converging to a strictly positive value whereas a semi-strong factor has mean-squared beta shrinking to zero at rate $n^{-2\alpha}$. Consider a nonrandom sequence $a_n = a^j n^{-2\delta}$ with $a > 0$, $0 < \delta < \alpha$. We allow the multiplicative constant a^j to vary across factors j, setting $a^j = (\frac{\sigma_r^2}{\sigma_{f,j}^2})/k$. This choice of a^j is motivated by the marginal R-squared shown above, see (27).

The sequence a_n distinguishes natural rate from semi-strong factors, in the sense that:

$$\begin{aligned} \lim_{n\to\infty} (\chi_j - a_n) &= \chi_j^* > 0 \text{ for all } j \leq k^{nr}, \text{ whereas} \\ \limsup_{n\to\infty} (\chi_j - a_n) &\leq 0 \text{ for all } j > k^{nr}. \end{aligned}$$

Once we have chosen the parameters of the sequence a_n, we compute the statistic:

$$z_j = n^{\frac{1}{2}}\left(\frac{\widehat{\chi}_j - a_n}{\psi_j^{\frac{1}{2}}}\right) \tag{28}$$

and note that for a natural rate factor $j \leq k^{nr}$ the statistic (28) has unit asymptotic power to reject $E[z_j] = 0$, whereas it has zero asymptotic power to reject $E[z_j] = 0$ for a semi-strong factor $j > k^{nr}$. The test statistic establishes a penalized benchmark value for the sum of squared betas as a function of n; if the estimated sum of squared betas is significantly above this benchmark, the factor is identified as a natural-rate factor.

4 Application to U.S. equity returns using Fama-French, currency and commodity factors

In this section we apply the methodology to analyze the currency and commodity exposures of US equity returns.

4.1 Data and Winsorization

We consider percentage changes in commodity prices and foreign exchange rates (against a currency basket) as semi-strong factors. We use daily returns on US equity returns over six periods, 1989-1993, 1994-1998, 1999-2003, 2004-2008, 2009-2013, and 2014-2018. In each subperiod we restrict the sample to equities with full return records over the 5-year subperiod. To moderate the influence of extreme return observations (which are often due to small, isolated trades of illiquid securities) we Winsorize the returns data at 99%; setting all observations in the top 0.5% and bottom 0.5% within each subperiod equal to the relevant boundary value. Table 1 shows the cross-sectional average of the first four time-series moments of returns in each of the subperiods, both before and after Winsorization. We also show the maximum and minimum returns in each subperiod; for the after-Winsorization case, these equal the Winsorization boundary values.

For currency returns, we use the U.S. Dollar, British Pound, Japanese Yen, and Korean Won, each measured against the global currency basket maintained by the International Monetary Fund (the Special Drawing Rights Index of the IMF). For commodities, we use the percentage changes in the prices of oil, gold, aluminum and forest products. The currency and commodity factors are obtained from Bloomberg. The daily factors are the percentage daily price changes. The codes in Bloomberg are: U.S. Dollar (XDRUSD), British Pound (XDRUSD divided by GBPUSD), Japanese Yen (USDJPY times XDRUSD), and Korean Won (USDKRW times XDRUSD), Brent crude futures traded on ICE (CO1), gold (XAU), aluminum (LMAHDY), random length lumber futures on the CME (LB1). In addition to the currency and commodity factors, we include the market, size and value factors from the Kenneth French data library.[3] Table 2 shows the first four time-series moments of the eleven factors for each of the six subperiods.

4.2 Estimation of mean and mean-square factor betas

Since the seemingly unrelated regression model has the same independent variables for all assets and no cross-equation restrictions, for the purpose of finding the cross-sectional averages of the coefficients (19), the set of n regressions "collapses" into a single time-series

[3] Available at: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

regression with an equally-weighted average of returns as the dependent variable. The results are shown in Table 3. The average asset betas for all three Fama-French factors are statistically significant in all subperiods (with t-statistics in excess of 45). The average asset betas relative to the currencies are significant in the post-2004 period, with the exception of the Japanese Yen. Additionally, the average betas relative to the Korean Won are significant in the second and third subperiods. The average betas for the Japanese Yen are only significant in the first subperiod, 1989-1993. The average betas for oil and aluminum are significant in all subperiods (aluminum) except one (oil). The average betas for lumber are generally insignificant. Table 3 clearly shows that the equity-return based Fama-French factors have a much stronger average relation to equity returns than the other factors.

To estimate mean-squared betas, we perform individual factor regressions for each of the securities in each subsample, and then compute the cross-sectional adjusted-mean-squared estimated betas shown in Table 4. To estimate the variances of the estimated mean-squared betas we repeat this full estimation process fifty thousand times with bootstrapped returns and compute the fifty-thousand-sample bootstrap variances for each mean-squared beta. In the latter three subperiods all of the factors except lumber have estimated mean-squared beta significantly greater than zero. Oil and gold are also significant in the other three, earlier, subperiods. The currency factors are mostly insignificant in these earlier three subperiods; aluminum also has mixed results in these earlier subperiods.

4.3 Distinguishing natural-rate from semi-strong factors

Table 5 shows the marginal R-squareds found by dropping one factor $j = 1, k$ from the stackedd regression model. Other than the three Fama-French factors, oil and gold show some non-negligible explanatory power by this measure. The stacked regression R-squared shows evidence of a secular increase from the earlier subperiods to the later ones.

Next we implement our test statistic (28) for distinguishing natural-rate from semi-strong factors. We set $\delta = \frac{1}{5}$ so that $n^{-2\delta}$ gives a reasonably small, but not excessively small, benchmark for mean-squared beta; for $k = 11$ and $n = 4250$, which is roughly the cross-sectional size in our subperiods, $n^{-2\delta}/k = 4250^{-(.4)}/11 = 0.0032$. Interpreted using (27), a factor must have a marginal R-squared above 0.3% to qualify as a natural-rate factor.

Table 6 shows the results of the statistical test for whether each factor has mean-square beta greater than $(\frac{\sigma_r^2}{\sigma_{f,j}^2})n^{-.4}/11$. The results in Table 6 provide strong support for the Fama-French factors being natural rate factors. The penalized mean-squared betas of the market factor MKTRF and size factor SMB are significantly positive in all subperiods; the value factor HML is significantly positive in four of the six subperiods. Oil and gold are significantly positive in two and three subperiods, respectively, thus showing mixed results as to whether they are best classified as natural-rate or semi-strong factors. We cannot reject the hypothesis that the remaining factors other than lumber are only semi-strong rather than natural rate factors (recall from Table 4 that lumber does not even qualify as a semi-strong factor).

The results in Tables 3, 4, and 6 suggest that the Fama-French factors are natural rate factors and that lumber is not a statistically identifiable semi-strong factor. Oil and gold are both significant factors, but give some conflicting results as to whether they should be considered natural rate or semi-strong factors. The remaining factors seem to be semi-strong but not natural-rate.

Lastly, to give an alternative perspective on the data, we use the Akaike Information Criterion (AIC) to choose factors with positive explanatory power by this criterion. To do this we must impose an assumption of normality on the asset-specific residuals in the stacked regression model (14), while still allowing heteroskedasticity across assets. We continue to assume independence of the asset specific returns across assets and across time. Under this normality assumption the ordinary least squares estimates are maximum likelihood, as is obvious. Including the unknown asset-specific return variances, the stacked regression model has $n(k+2)$ parameters and nT observations, so the AIC criterion is:

$$AIC = 2n(k+2) - 2\log(\widehat{L}) \tag{29}$$

where $\widehat{L}$ is the log likelihood, which in this stacked linear regression model is

$$\widehat{L} = -\frac{nT}{2}\log(2\pi) - \frac{T}{2}\sum_{i=1}^{n}\log(\widehat{\sigma}_{\varepsilon i}^2) - \frac{n}{2}.$$

Akaike's approach is different from our earlier approach, in that it treats different specifications (in our case, models with more or fewer factors) as competing approximate models of the data, rather than as nested models that are the true data generating process. Although it does not fit exactly into our modeling framework we believe it gives a useful alternative perspective on the data. (See also Bai and Ng (2002) who use an AIC-type metric to choose natural-rate factors in a statistical factor model.) Table 7 shows the results from stepwise regression based on (29). The three Fama-French factors are always included in the final model. Gold is included in the final model in all six subperiods, and oil in five of the six. The results are mixed for aluminum and the four foreign exchange factors; they are dropped in the earlier subperiods, but kept in the later ones. Lumber is the only factor to be dropped in all six subperiods.

5 Summary

This paper develops and implements a new econometric methodology for estimating the cross-sectional influence of "small" factors (that is, factors with only modest explanatory power) in a large cross-section of asset returns. We build upon the approximate factor modeling framework of Chamberlain and Rothschild (1983) in which pervasive factors are identified by the unbounded growth of their sum of squared factor betas as the number of assets n grows large. Following Chudik et al. (2011), we differentiate between natural-rate factors, whose sum of squared betas grow at rate n, and semi-strong factors, whose sum of squared betas grow at a slower rate. We provide a unified framework by embedding a factor model with both natural-rate and semi-strong factors into a random coefficients framework. We develop an estimation methodology based on the large$-n$ statistical distributions of the cross-sectional mean and mean-square of time-series regression-estimated factor betas. In asymptotically large cross-sections, in order to be statistically identified the mean or mean-squared beta must have a magnitude declining at a slower rate than the standard deviation of estimation, which declines at rate square-root-n.

We apply the methodology to the daily returns of US equity returns using Fama and French's market, size, and value factors, and four currency (US dollar, British pound, Japanese Yen, and Korean Won) and four commodity (oil, gold, aluminum and lumber) factors. We divide our 1989-2018 database into six five-year subperiods. The three Fama-French factors qualify as natural-rate factors. Oil and gold are both significant factors but give some conflicting results as to whether they are natural-rate or semi-strong factors. Aluminum and the four currency factors are semi-strong factors in the more recent subperiods, but not natural-rate factors. Lumber does not qualify as either a natural-rate or semi-strong factor.

Appendix 1: The Sample Variance of the Squared Regression Coefficient

This appendix derives the sample variance of the adjusted squared ordinary least squares regression coefficient within the context of our model. Recall the expression (15) for the time-series estimated regression coefficient:

$$\widehat{B}_{ij} = B_{ij} + \sum_{t=1}^{T} m_{jt}\varepsilon_{it} \tag{A1}$$

and the adjusted square of this estimate $(\widehat{B}_{ij})^2 - [(F'F)^{-1}]_{j+1,j+1}\widehat{\sigma}^2_{\varepsilon_i}$. Note that $[(F'F)^{-1}]_{j+1,j+1} = \overline{m}_{2j}$ (to see this, write $(F'F)^{-1} = (F'F)^{-1}F'F(F'F)^{-1}$ and note that $\overline{m}_{2j}$ is the $j+1^{th}$ diagonal entry). It is easy to show that $E[\widehat{B}^2_{ij} - \overline{m}_{2j}\widehat{\sigma}^2_{\varepsilon_i}] = B^2_{ij}$. Now we derive the variance of this estimator, that is:

$$Var[(\widehat{B}_{ij})^2 - \overline{m}_{2j}\widehat{\sigma}^2_{\varepsilon_i}] = Var[(\widehat{B}_{ij})^2] + (\overline{m}_{2j})^2 Var[\widehat{\sigma}^2_{\varepsilon_i}] - 2\overline{m}_{2j}Cov[(\widehat{B}_{ij})^2, \widehat{\sigma}^2_{\varepsilon_i}]. \tag{A2}$$

We compute each of the three components on the right-hand side of (A2) separately, beginning with the first. Note that:

$$Var[(\widehat{B}_{ij})^2] = E[((\widehat{B}_{ij})^2 - \overline{m}_{2j}\sigma^2_{\varepsilon_i} - B^2_{ij})^2]. \tag{A3}$$

Expressing $(\widehat{B}_{ij})^2$ using (A1) :

$$(\widehat{B}_{ij})^2 = B^2_{ij} + 2B_{ij}\sum_{t=1}^{T} m_{jt}\varepsilon_{it} + \sum_{t=1}^{T}\sum_{\tau=1}^{T} m_{jt}m_{j\tau}\varepsilon_{it}\varepsilon_{i\tau}. \tag{A4}$$

Inserting (A4) into the right-hand side of (A3) gives:

$$E[((\widehat{B}_{ij})^2 - \overline{m}_{2j}\sigma^2_{\varepsilon_i} - B^2_{ij})^2] = E[(2B_{ij}\sum_{t=1}^{T} m_{jt}\varepsilon_{it} + \sum_{t=1}^{T}\sum_{\tau=1}^{T} m_{jt}m_{j\tau}\varepsilon_{it}\varepsilon_{i\tau} - \overline{m}_{2j}\sigma^2_{\varepsilon_i})^2]. \tag{A5}$$

Expanding out the square on the right-hand side of (A5) gives three square terms and three cross terms:

$$
\begin{aligned}
& E[(2B_{ij}\sum_{t=1}^{T}m_{jt}\varepsilon_{it} + \sum_{t=1}^{T}\sum_{\tau=1}^{T}m_{jt}m_{j\tau}\varepsilon_{it}\varepsilon_{i\tau} - \overline{m}_{2j}\sigma^2_{\varepsilon_i})^2] \\
= \;& 4B_{ij}^2\sum_{t=1}^{T}\sum_{\tau=1}^{T}m_{jt}m_{j\tau}\varepsilon_{it}\varepsilon_{i\tau} + && \text{(A6)} \\
& \sum_{t=1}^{T}\sum_{\tau=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}m_{jt}m_{j\tau}m_{js}m_{ju}\varepsilon_{it}\varepsilon_{i\tau}\varepsilon_{is}\varepsilon_{iu} + && \text{(A7)} \\
& (\overline{m}_{2j}\sigma^2_{\varepsilon_i})^2 + && \text{(A8)} \\
& 4B_{ij}\sum_{t=1}^{T}\sum_{\tau=1}^{T}\sum_{s=1}^{T}m_{jt}m_{j\tau}m_{js}\varepsilon_{it}\varepsilon_{i\tau}\varepsilon_{is} - && \text{(A9)} \\
& 4B_{ij}\overline{m}_{2j}\sigma^2_{\varepsilon_i}\sum_{t=1}^{T}m_{jt}\varepsilon_{it} - && \text{(A10)} \\
& 2\overline{m}_2\sigma^2_{\varepsilon_i}\sum_{t=1}^{T}\sum_{\tau=1}^{T}m_{jt}m_{j\tau}\varepsilon_{it}\varepsilon_{i\tau}. && \text{(A11)}
\end{aligned}
$$

We will consider each of the six additive components above in order. For (A6), all the cross -terms $E[\varepsilon_{it}\varepsilon_{i\tau}]$, $t \neq \tau$ have expectation of zero and all the pure terms $E[\varepsilon_{it}^2]$ have expectation $\sigma^2_{\varepsilon_i}$, giving

$$
E[4B_{ij}^2\sum_{t=1}^{T}\sum_{\tau=1}^{T}m_{jt}m_{j\tau}\varepsilon_{it}\varepsilon_{i\tau}] = 4B_{ij}^2\overline{m}_{2j}\sigma^2_{\varepsilon_i}.
$$

For (A7), there is one pure term where all four time indices are equal and three cross-product terms where two pairs of time indices are equal:

$$
E[\sum_{t=1}^{T}\sum_{\tau=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}m_{jt}m_{j\tau}m_{js}m_{ju}\varepsilon_{it}\varepsilon_{i\tau}\varepsilon_{is}\varepsilon_{iu}] = \overline{m}_{4j}E[\varepsilon_i^4] + 3\overline{m}_{22j}(\sigma^2_{\varepsilon_i})^2.
$$

(A8) is in a simple form already. For (A9), the only nonzero expectation is the pure sum when all three time indices are equal:

$$E[4B_{ij}\sum_{t=1}^{T}\sum_{\tau=1}^{T}\sum_{s=1}^{T}m_{jt}m_{j\tau}m_{js}\varepsilon_{it}\varepsilon_{i\tau}\varepsilon_{is}] = 4B_{ij}\overline{m}_{3j}E[\varepsilon_i^3].$$

Term (A10) has expectation zero. For (A11):

$$E[2\overline{m}_{2j}\sigma_{\varepsilon_i}^2\sum_{t=1}^{T}\sum_{\tau=1}^{T}m_{jt}m_{j\tau}\varepsilon_{it}\varepsilon_{i\tau}] = 2(\overline{m}_{2j}\sigma_{\varepsilon_i}^2)^2.$$

Adding together the components:

$$\begin{aligned} Var[(\widehat{B}_{ij})^2] &= 4B_{ij}^2\overline{m}_{2j}\sigma_{\varepsilon_i}^2 + \overline{m}_{4j}E[\varepsilon_i^4] + 3\overline{m}_{22j}(\sigma_{\varepsilon_i}^2)^2 + (\overline{m}_{2j}\sigma_{\varepsilon_i}^2)^2 + \\ &\quad 4B_{ij}\overline{m}_{3j}E[\varepsilon_i^3] - 2(\overline{m}_{2j}\sigma_{\varepsilon_i}^2)^2 \\ &= 4\overline{m}_{2j}B_{ij}^2\sigma_{\varepsilon_i}^2 + 4B_{ij}\overline{m}_{3j}E[\varepsilon_i^3] + \overline{m}_{4j}E[\varepsilon_i^4] + (3\overline{m}_{22j} - (\overline{m}_{2j})^2)(\sigma_{\varepsilon_i}^2)^2 \quad \text{(A12)} \end{aligned}$$

Next we derive the second term of (A2). Recall that $\widehat{\sigma}_{\varepsilon_i}^2 = \sum_{t=1}^{T}\sum_{\tau=1}^{T}p_{t\tau}\varepsilon_{it}\varepsilon_{i\tau}$. Taking the expectation of the square minus the squared expectation:

$$Var[\widehat{\sigma}_{\varepsilon_i}^2] = E[\sum_{t=1}^{T}\sum_{\tau=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}p_{t\tau}p_{su}\varepsilon_{it}\varepsilon_{i\tau}\varepsilon_{is}\varepsilon_{iu}] - (\sigma_{\varepsilon_i}^2)^2,$$

and using that ε_{it} is independent across time with zero mean:

$$\begin{aligned} &= E[\varepsilon_i^4]\sum_{t=1}^{T}p_{tt}^2 + (\sigma_{\varepsilon_i}^2)^2(\sum_{t=1}^{T}\sum_{\tau\neq t}2p_{t\tau}^2 + p_{tt}p_{\tau\tau}) - (\sigma_{\varepsilon_i}^2)^2 \\ &= \overline{p}_2E[\varepsilon_i^4] + ((p\times p) - 1)(\sigma_{\varepsilon_i}^2)^2 \quad \text{(A13)} \end{aligned}$$

The last term in (A2) involves the covariance between $(\widehat{B}_{ij})^2$ and $\widehat{\sigma}_{\varepsilon_i}^2$. Taking the product of their de-meaned values:

$$Cov[(\widehat{B}_{ij})^2, \widehat{\sigma}_{\varepsilon_i}^2] = E[(\sum_{t=1}^{T}\sum_{\tau=1}^{T}p_{t\tau}\varepsilon_{it}\varepsilon_{i\tau} - \sigma_{\varepsilon_i}^2)(2B_{ij}\sum_{t=1}^{T}m_{jt}\varepsilon_{it} + \sum_{t=1}^{T}\sum_{\tau=1}^{T}m_{jt}m_{j\tau}\varepsilon_{it}\varepsilon_{i\tau})]. \quad \text{(A14)}$$

Expanding out the four terms in (A14):

$$
\begin{aligned}
= \;& 2B_{ij}E[\sum_{t=1}^{T}\sum_{\tau=1}^{T}\sum_{s=1}^{T}p_{t\tau}m_{js}\varepsilon_{it}\varepsilon_{i\tau}\varepsilon_{is}] + \\
& E[\sum_{t=1}^{T}\sum_{\tau=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}p_{t\tau}m_{js}m_{ju}\varepsilon_{it}\varepsilon_{i\tau}\varepsilon_{is}\varepsilon_{iu}] \\
& -\sigma^2_{\varepsilon_i}2B_{ij}E[\sum_{t=1}^{T}m_{jt}\varepsilon_{it}] - \sigma^2_{\varepsilon_i}E[\sum_{t=1}^{T}\sum_{\tau=1}^{T}m_{jt}m_{j\tau}\varepsilon_{it}\varepsilon_{i\tau}].
\end{aligned}
$$

and then using the ε_{it} is independent through time with zero mean:

$$
\begin{aligned}
= \;& 2B_{ij}E[\varepsilon_i^3]\sum_{t=1}^{T}p_{tt}m_{jt} + E[\varepsilon_i^4]\sum_{t=1}^{T}p_{tt}m_{jt}^2 \\
& +(\sigma^2_{\varepsilon_i})^2\sum_{t=1}^{T}\sum_{\tau\neq t}(2p_{t\tau}m_{jt}m_{j\tau} + p_{tt}m_{j\tau}^2) - (\sigma^2_{\varepsilon_i})^2\sum_{t=1}^{T}m_{jt}^2 \\
= \;& 2B_{ij}\overline{pm}_jE[\varepsilon_i^3] + \overline{pm}_{2j}E[\varepsilon_i^4] + ((p\times m) - \overline{m}_{2j})(\sigma^2_{\varepsilon_i})^2. \qquad \text{(A15)}
\end{aligned}
$$

Now we collect the terms from (A12), (A13), and (A15) and combine them into (A1), multiplying (A13) terms by $(\overline{m}_{2j})^2$ and (A15) terms by $-2\overline{m}_{2j}$, giving:

$$
\begin{aligned}
Var[(\widehat{B}_{ij})^2 - \overline{m}_{j2}\widehat{\sigma}^2_{\varepsilon_i}] \;=\;& \\
& 4\overline{m}_{2j}B_{ij}^2\sigma^2_{\varepsilon_i} + 4B_{ij}\overline{m}_{3j}E[\varepsilon_i^3] + \overline{m}_{4j}E[\varepsilon_i^4] + (3\overline{m}_{22j} - (\overline{m}_{2j})^2)(\sigma^2_{\varepsilon_i})^2 \\
& +(\overline{m}_{2j})^2(\overline{p}_2E[\varepsilon_i^4] + ((p\times p) - 1)(\sigma^2_{\varepsilon_i})^2) \\
& -2\overline{m}_{2j}(2B_{ij}\overline{pm}_jE[\varepsilon_i^3] + \overline{pm}_{2j}E[\varepsilon_i^4] + ((p\times m) - \overline{m}_{2j})(\sigma^2_{\varepsilon_i})^2)).
\end{aligned}
$$

Sorting by the moments and simplifying, this becomes:

$$\begin{aligned}
Var[(\widehat{B}_{ij})^2 - \overline{m}_{2j}\widehat{\sigma}^2_{\varepsilon_i}] &= c_{1ij}\sigma^2_{\varepsilon_i} + c_{2ij}E[\varepsilon_i^3] + c_{3j}E[\varepsilon_i^4] + c_{4j}(\sigma^2_{\varepsilon_i})^2 \\
c_{1ij} &= 4\overline{m}_{2j}B^2_{ij} \\
c_{2ij} &= 4B_{ij}\overline{m}_{3j} - 4\overline{m}_{2j}B_{ij}(\overline{pm}_j) \\
c_{3j} &= \overline{m}_{4j} + (\overline{m}_{2j})^2\overline{p}_2 - 2\overline{m}_{2j}(\overline{pm}_{2j}) \\
c_{4j} &= 3\overline{m}_{22j} + (\overline{m}_{2j})^2(p \times p) - 2\overline{m}_{2j}(p \times m)
\end{aligned}$$

Appendix 2: Discussion of the Bootstrap Estimation of the Chi Statistic's Estimation Variance

This appendix discusses the application of random-case bootstrapping to our model. We use the technique to estimate the variance of our chi-statistic (23).

We have derived an exact formula for the variance of the chi-statistic under the assumptions of the model, but it is a function of the third and fourth moments of asset-specific returns, which are difficult to estimate consistently from time-series regression residuals. The boostrapping approach to estimating $var[\widehat{\chi}_j]$ is natural in this context. Bootstrapping uses the full available sample to estimate this variance directly from the sample data. To implement this estimator we act as if the time series i.i..d. multivariate probability distribution generating returns and factors consists of an $n+k$-vector process, s_τ which has T discrete possible realizations

$$s_\tau = [vec[R_\tau], vec[F_\tau]].$$

The discrete set of potential state realizations for s_τ are assumed equal to our observed sample values, and each of these T random states is assumed to have equal probability $\frac{1}{T}$. This is fully consistent with all of our factor model assumptions on returns.

Within the context of this discretized probability distribution we can derive the exact variance of the chi-statistic. We note that the chi-statistic is a function of a sample of T random realizations of the state process:

$$\widehat{\chi}_j = f(s_1, ..., s_T)$$

where $s_1, ..., s_T$ are independently and identically distributed realizations, and the formula for $f(\cdot)$ is the estimation formula for $\widehat{\chi}_j$. Using the discretized probability distribution, the exact variance of $\widehat{\chi}_j$ is:

$$var[\widehat{\chi}_j] = \frac{1}{T^T}\sum_{S^T} f(s_1, ..., s_T)^2 - (\frac{1}{T^T}\sum_{S^T} f(s_1, ..., s_T))^2 \tag{30}$$

where the averages run across the set of all $T-$tuples $s_1, ..., s_T$. Computing (30) is straightforward but impractical since with $T = 1200$ (roughly our sample size) there are 1200^{1200} terms in the averages. However the averages are easily and closely approximated by simply averaging over a large number of random draws of $(s_1, ..., s_T)$; we use $50,000$ draws in each subperiod.

REFERENCES

Asness, C., A. Frazzini, and L. H. Pedersen (2017): "Quality Minus Junk," Working Paper.

Burnside, C. (2016): "Identification and Inference in Linear Stochastic Discount Factor Models," Journal of Financial Econometrics, **14**(2), 296–330.

Bai, Jushan and Serena Ng. (2002) "Determining the Number of Factors in Approximate Factor Models," *Econometrica* **70**, 191–221.

Bai, Jushan, and Serena Ng. (2008) "Large Dimensional Factor Analysis," *Foundations and Trends in Econometrics* **3**, 89–163.

Boivin, Jean, and Serena Ng. (2006) "Are More Data Always Better for Factor Analysis?" *Journal of Econometrics* **132**, 169−194.

Bryzgalova, Svetlana, (2016) "Spurious Factors in Linear Asset Pricing Models," working paper, Stanford University, February.

Chamberlain, Gary, and Michael Rothschild, (1983) "Arbitrage, Factor Structure and Mean-Variance Analysis in Large Asset Markets," *Econometrica* **51**, 1305–1324.

Choi, In, (2012) "Efficient Estimation of Factor Models," *Econometric Theory* **28**, 274–308.

Chudik, Alexander, M. Hashem Pesaran and Elisa Tosetti (2011) "Weak and strong cross-section dependence and estimation of large panels," *Econometrics Journal*, **14**, C45–C90.

Connor, Gregory, and Robert A. Korajczyk, (1986) "Performance Measurement with the

Arbitrage Pricing Theory: A New Framework for Analysis," *Journal of Financial Economics* **15**, 323−346.

Connor, Gregory, and Robert A. Korajczyk, (1988) "Risk and Return in an Equilibrium APT: Application of a New Test Methodology," *Journal of Financial Economics* **21**, 255−290.

Connor, Gregory, Robert A. Korajczyk and Robert Uhlaner, (2015), A Synthesis of Two Factor Estimation Methods, *Journal of Financial and Quantitative Analysis*, **50**(4), 825–842.

Davidson, Russel, (1986), *Stochastic Limit Theory*, Oxford University Press, Oxford, United Kingdom.

Davidson, Russel and James G. MacKinnon, (1993), *Estimation and Inference in Econometrics*, by Oxford University Press, Oxford, U.K.

Dong, Chaohua, Jiti Gao and Bin Peng (2018), "Varying Coefficient Panel Data Models with Partially Observed Factor Structure," working paper, University of Bath, January.

Fama, Eugene F., and Kenneth R. French, (1993) "Common Risk Factors in the Returns on Stocks and Bonds," *Journal of Financial Economics* **33**, 5−56.

Fama, E. F. and K. R. French (2016): "Dissecting Anomalies with a Five-Factor Model," *Review of Financial Studies*, vol. 29, 69–103.

Forni, Mario, Marc Hallin, Marco Lippi, and Lucrezia Reichlin, (2005) "The Generalized Dynamic Factor Model: One-Sided Estimation and Forecasting," *Journal of the American Statistical Association* **100,** 830–840.

Gagliardini, Patrick, and Christian Gourieroux, (2014) "Efficiency in Large Dynamic

Panel Models with Common Factor," *Econometric Theory*, **30**, 961–1020.

Gospodinov, N., R. M. Kan, and C. Robotti (2014) "Misspecification-Robust Inference in Linear Asset Pricing Models with Irrelevant Risk Factors," *Review of Financial Studies*, **27**, 2139–2170.

——— (2017): "Spurious Inference in Reduced-Rank Asset-Pricing Models," *Econometrica* **85**, 1613–1628.

Harvey, C., Y. Liu, and H. Zhu (2016) "...and the Cross-Section of Expected Returns," *Review of Financial Studies*, **29**(1), 5–68.

Haugen, R. and N. Baker (1996): "Commonality in the Determinants of Expected Stock Returns," *Journal of Financial Economics*, 41, 401–439.

Hou, K., C. Xue, and L. Zhang (2015): "Digesting Anomalies: An Investment Approach," *Review of Financial Studies*, **28**(3), 650–705.

Jones, Christopher S., (2001) "Extracting Factors from Heteroskedastic Asset Returns," *Journal of Financial Economics* **62**, 293−325.

Kan, R. M. and C. Zhang (1999a): "GMM Tests of Stochastic Discount Factor Models with Useless Factors," *Journal of Financial Economics*, 54, 103–27.

——— (1999b): "Two-pass Tests of Asset Pricing Models with Useless Factors," *Journal of Finance*, 54, 204–235.

Kao, Chihwa, Lorenzo Trapani and Giovanni Urga, (2012) "Asymptotics for Panel Models with Common Shocks," *Econometric Reviews* **31,** 390-439.

Kleibergen, F. (2009): "Tests of Risk Premia in Linear Factor Models," Journal of Econometrics, **149**,149–173.

Kleibergen, F. and Z. Zhan (2015): “Unexplained Factors and their Effects on Second Pass R-squareds and t-Tests,” *Journal of Econometrics* **189**(1), 101–116.

Kroencke, T. A. (2017) “Asset Pricing Without Garbage,” *Journal of Finance* **72**(1), 47–98.

Lewellen, J., S. Nagel, and J. Shanken (2010): “A skeptical appraisal of asset pricing tests,” *Journal of Financial Economics*, 96, 175–194.

Lustig, H., N. Roussanov, and A. Verdelhan (2011): “Common Risk Factors in Currency Markets,” *Review of Financial Studies*, 3731–77.

Lustig, H. and A. Verdelhan (2007): “The Cross-Section of Foreign Currency Risk Premia and US Consumption Growth Risk,” *American Economic Review*, **97**(1), 89–117.

MacKinlay, Andrew. C., and Lubos Pastor (2000) "Asset Pricing Models: Implications for Expected Returns and Portfolio Selection," Review of Financial Studies, 13(4), 883–916.

Onatski, Alexei (2015) "Asymptotic analysis of the squared estimation error in misspecified factor models," *Journal of Econometrics* **186**, 388–406.

——————— (2018) "Asymptotics of the Principal Components Estimator of Large Factor Models with Weak Factors and I.I.D. Gaussian Noise," Cambridge Working Paper Economics no. 1808, January.

Novy-Marx, R. (2014): “Predicting Anomaly Performance with Politics, the Weather, Global Warming, Sunspots,and the Stars,” *Journal of Financial Economics*, **112**(2), 137-146.

Reisman, Haim (1992) "Reference Variables, Factor Structure, and the Approximate Multibeta Representation," *Journal of Finance*, **47**(4), 1303–1314.

Ross, Stephen A., (1976) "The arbitrage theory of capital asset pricing," *Journal of Economic Theory* **13**, 341–360.

Shanken, Jay (1992): "On the Estimation of Beta-Pricing Models," *The Review of Financial Studies*, **5**, 1–33.

Stock, James H., and Mark W. Watson, (2002) "Forecasting Using Principal Components From a Large Number of Predictors," *Journal of the American Statistical Association, Theory and Methods* **97**, 1−13.

Uppal, Raman and Paolo Zaffaroni (2016), "Portfolio Choice with Model Misspecification: A Foundation for Alpha and Beta Portfolios," working paper, Imperial College Business School, July.

White, Halbert, (1984) *Asymptotic Theory for Econometricians*, Academic Press, London.

Table 1: Cross-sectional averages of time-series moments of daily returns before and after Winsorization

		Mean	variance	skewness	kurtosis	minimum	maximum	n; T
1989-1993	before Winsorization	0.00106	0.00200	2.536	66.411	-87.51%	483.32%	4254; 1265
	after Winsorization	0.00106	0.00156	0.672	11.471	-18.75%	23.08%	
1999-2003	before Winsorization	0.00076	0.00163	2.1913	61.048	-86.02%	585.70%	4817; 1263
	after Winsorization	0.00081	0.00133	0.63243	10.194	-16.67%	20.83%	
1999-2003	before Winsorization	0.00097	0.00186	2.6552	80.335	-95.66%	1595.50%	4656; 1256
	after Winsorization	0.00095	0.00143	0.77538	10.919	-17.46%	22.92%	
	before Winsorization	0.00005	0.00102	2.6821	100.94	-89.64%	600.00%	4313; 1259
2004-2008	after Winsorization	0.00003	0.00072	0.53839	10.625	-12.90%	14.54%	
	before Winsorization	0.00106	0.00090	3.6146	134.5	-82.96%	360.49%	4795; 1258
2009-2013	after Winsorization	0.00092	0.00064	0.6271	11.236	-11.67%	14.29%	
	before Winsorization	0.00020	0.00069	4.2352	210.85	-92.28%	677.49%	4904; 1258
2014-2018	after Winsorization	0.00011	0.00043	0.33897	10.683	-9.93%	11.11%	

Notes on Table 1: The table shows the first four moments of returns and the minimum and maximum return within each of the six five-year subperiods, before and after Winsorization at the 0.5% and 99.5% fractiles. The last column shows the number of cross-sectional (n) and time-series (T) observations. CRSP daily returns in excess of the risk-free rate for all securities with full five-year return histories.

Table 2: First four moments of time-series factors

		MKTRF	SMB	HML	USDSDR	GBPSDR	JPYSDR	KRWSDR	Oil	Gold	Alum	Lumber
1989-1993	mean	0.00035	-0.00004	0.00009	0.00003	0.00019	-0.00004	0.00017	0.00018	0.00001	-0.00055	0.00093
	variance	0.00005	0.00002	0.00001	0.00002	0.00006	0.00004	0.00002	0.00056	0.00007	0.00022	0.00033
	skewness	-0.518	-0.496	0.463	0.088	0.008	0.091	-0.022	-2.376	-0.942	0.189	0.904
	kurtosis	7.610	6.257	6.514	6.078	4.688	6.011	7.524	45.501	11.979	7.925	17.818
1999-2003	mean	0.00063	-0.00032	0.00010	0.00003	-0.00010	0.00005	0.00034	0.00002	-0.00022	0.00011	-0.00007
	variance	0.00007	0.00003	0.00002	0.00001	0.00003	0.00006	0.00022	0.00039	0.00004	0.00015	0.00057
	skewness	-0.759	-0.041	0.060	0.577	0.293	-0.605	1.203	0.040	-0.115	-0.308	0.429
	kurtosis	11.485	4.458	6.006	7.589	5.549	7.398	55.386	7.618	7.224	7.000	10.484
1999-2003	mean	-0.00002	0.00040	0.00031	0.00004	-0.00002	0.00000	0.00003	0.00103	0.00033	0.00026	0.00030
	variance	0.00018	0.00006	0.00006	0.00001	0.00003	0.00004	0.00003	0.00055	0.00009	0.00010	0.00047
	skewness	0.140	-0.734	0.060	0.284	0.044	0.002	0.390	-0.310	1.349	0.299	0.654
	kurtosis	4.303	7.092	5.399	4.520	3.374	4.598	4.364	4.999	14.951	4.472	7.386
2004-2008	mean	-0.00010	0.00001	0.00015	0.00003	0.00018	-0.00008	0.00010	0.00072	0.00067	0.00009	-0.00025
	variance	0.00018	0.00004	0.00004	0.00001	0.00003	0.00005	0.00007	0.00054	0.00018	0.00025	0.00048
	skewness	-0.077	-0.161	1.176	-0.143	0.511	0.072	-0.463	0.084	-0.058	-0.387	1.322
	kurtosis	17.675	9.384	20.043	6.141	7.883	10.465	62.219	6.143	8.841	4.885	14.892
2009-2013	mean	0.00076	0.00017	-0.00002	0.00001	-0.00014	0.00016	-0.00014	0.00098	0.00026	0.00032	0.00089
	variance	0.00016	0.00003	0.00005	0.00001	0.00003	0.00005	0.00005	0.00035	0.00014	0.00024	0.00060
	skewness	-0.179	0.166	0.236	0.186	0.074	0.084	0.460	0.007	-0.641	-0.027	1.448
	kurtosis	6.881	4.806	10.807	7.157	4.275	4.916	10.118	6.265	7.201	4.473	11.694
2014-2018	mean	0.00032	-0.00009	-0.00011	-0.00008	0.00013	-0.00004	-0.00004	-0.00020	0.00008	0.00014	0.00012
	variance	0.00007	0.00003	0.00003	0.00001	0.00003	0.00003	0.00002	0.00047	0.00007	0.00014	0.00034
	skewness	-0.44598	0.19017	0.54189	-0.40807	1.4283	-0.71277	0.03695	0.32194	0.31169	0.33381	-0.17648
	kurtosis	6.4183	3.842	4.7662	7.5632	20.953	10.716	3.5675	5.6263	5.5637	6.0374	7.8895

Notes on Table 2: The table shows the first four moments of the eleven factors within each of the six five-year subperiods. MKTRF, SMB, and HML are the Fama-French market, size and value factors; USDSDR, GBPSDR, JPYSDR and KRWSDR are the percentage change in the US dollar, British Pound Sterling, Japanese Yen, and Korean Won in units of IMF Special Drawing Rights; Oil, Gold, Aluminum and Lumber are the percentage changes in the prices of these commodities. See text for details of data sources.

Table 3: Cross-sectional average factor betas and their t-statistics

	MKTRF	SMB	HML	USDSDR	GBPSDR	JPYSDR	KRWSDR	Oil	Gold	Alum	Lumber	Rbar2
1989-1993	**0.803**	**0.671**	**0.225**	-0.009	-0.003	**-0.016**	0.004	-0.006	0.007	**0.005**	0.003	94.42%
	126.835	**75.835**	**20.410**	-0.506	-0.619	**-2.725**	0.290	-4.127	1.749	**2.098**	1.654	
1999-2003	**0.843**	**0.659**	**0.291**	0.006	0.000	-0.002	**-0.006**	0.004	0.009	**0.007**	0.003	95.35%
	126.631	**77.063**	**22.969**	0.423	0.046	-0.463	**-2.503**	1.890	1.415	**2.300**	1.820	
1999-2003	**0.726**	**0.446**	**0.337**	0.034	-0.011	-0.015	**-0.063**	**0.007**	0.013	**0.025**	**0.010**	93.16%
	105.838	**45.254**	**27.443**	1.287	-0.764	-1.397	**-4.371**	**2.608**	1.811	**3.788**	**3.166**	
	0.790	**0.449**	**0.158**	**0.071**	**-0.030**	0.003	**-0.083**	**0.021**	0.006	**0.022**	0.005	97.31%
2004-2008	**157.293**	**48.342**	**17.563**	**3.697**	**-2.858**	0.274	**-12.389**	**7.950**	1.186	**5.898**	1.857	
2009-2013	**0.805**	**0.410**	**0.159**	**0.105**	**-0.041**	-0.002	**-0.045**	**0.019**	**0.021**	**0.012**	0.004	98.30%
	141.050	**45.772**	**19.549**	**6.103**	**-4.555**	-0.226	**-6.462**	**6.180**	**5.036**	**3.314**	1.911	
2014-2018	**0.733**	**0.417**	**0.129**	**0.066**	**-0.043**	-0.003	**-0.062**	**0.030**	**0.054**	**0.011**	0.003	96.51%
	132.515	**54.598**	**15.692**	**3.772**	**-6.152**	-0.318	**-7.681**	**14.665**	**9.614**	**3.160**	1.434	

Notes on Table 3: The table shows the regression coefficients (first row in each subperiod) and their t-statistics (second row in each subperiod) from the regression of the equally-weighted asset return on the eleven factors and an intercept. Numbers in bold are significantly different from zero with 95% confidence.

Table 4: Adjusted mean-squared beta estimates and their z-statistics

	MKTRF	SMB	HML	USDSDR	GBPSDR	JPYSDR	KRWSDR	Oil	Gold	Alum	Lumber
1989-1993	**0.844**	**0.647**	**0.164**	0.035	0.005	0.004	0.025	**0.001**	**0.019**	-0.001	0.000
	59.683	**28.359**	**7.060**	0.367	1.550	0.884	0.266	**1.921**	**4.976**	-0.612	-0.274
1994-1998	**0.873**	**0.627**	**0.259**	0.003	0.000	0.004	-0.001	**0.001**	**0.040**	0.001	0.000
	48.788	**26.927**	**12.156**	0.166	-0.070	1.279	-0.433	**1.947**	**7.404**	0.615	0.109
1999-2003	**0.753**	**0.422**	**0.310**	0.011	0.005	0.004	**0.023**	**0.003**	**0.023**	**0.003**	0.000
	50.787	**18.841**	**16.105**	0.490	0.888	0.933	**2.149**	**5.062**	**5.753**	**1.681**	0.126
2004-2008	**0.791**	**0.469**	**0.183**	**0.064**	**0.022**	**0.017**	**0.038**	**0.009**	**0.021**	**0.004**	0.000
	56.175	**23.876**	**13.057**	**2.569**	**2.807**	**2.753**	**2.599**	**8.575**	**7.270**	**2.844**	1.015
2009-2013	**0.887**	**0.431**	**0.189**	**0.101**	**0.022**	**0.008**	**0.020**	**0.007**	**0.036**	**0.002**	0.000
	51.236	**32.853**	**15.165**	**4.696**	**4.209**	**2.787**	**3.271**	**6.648**	**12.816**	**3.001**	1.109
2014-2018	**0.803**	**0.441**	**0.235**	**0.050**	**0.016**	**0.015**	**0.017**	**0.016**	**0.113**	**0.002**	0.000
	55.153	**39.644**	**26.888**	**3.550**	**4.451**	**4.075**	**4.151**	**13.434**	**16.428**	**2.979**	-0.020

Notes on Table 4: The table shows the adjusted mean-squared beta estimates (first row in each subperiod) and their z-statistics (second row in each subperiod) for the test that the mean-squared beta of the factor is greater than zero. Numbers in bold are significantly greater than zero with 95% confidence.

Table 5: Panel regression marginal R2’s for each factor

	MKTRF	SMB	HML	USDSDR	GBPSDR	JPYSDR	KRWSDR	Oil	Gold	Alum	Lumber	Stacked R-squared
1989-1993	1.511%	0.594%	0.097%	0.008%	0.016%	0.009%	0.008%	0.035%	0.080%	-0.007%	-0.004%	2.640%
1994-1998	1.996%	0.868%	0.164%	0.002%	-0.001%	0.015%	-0.009%	0.022%	0.098%	0.006%	0.001%	4.109%
1999-2003	4.569%	1.237%	0.589%	0.004%	0.007%	0.011%	0.031%	0.090%	0.128%	0.018%	0.001%	9.552%
2004-2008	10.670%	1.850%	0.769%	0.059%	0.065%	0.065%	0.288%	0.441%	0.295%	0.085%	0.016%	22.726%
2009-2013	7.175%	1.414%	0.753%	0.091%	0.071%	0.036%	0.109%	0.199%	0.535%	0.053%	0.021%	28.821%
2014-2018	8.144%	2.342%	1.075%	0.051%	0.100%	0.058%	0.080%	1.226%	1.103%	0.048%	0.000%	21.674%

Notes on Table 5: The table shows the marginal R-squared found by deleting each factor individually from the stacked regression model. The last column is the R-squared of the stacked regression model; see the text for the precise formulas.

Table 6: Test for natural rate versus semi-strong factors (z-statistics)

	MKTRF	SMB	HML	USDSDR	GBPSDR	JPYSDR	KRWSDR	Oil	Gold	Alum	Lumber
1989-1993	**53.000**	**17.025**	-8.847	-2.981	-25.246	-25.791	-2.364	-14.527	-13.513	-27.095	-23.122
1994-1998	**45.576**	**20.222**	0.681	-22.739	-24.545	-20.926	-14.473	-24.635	-13.166	-31.582	-25.382
1999-2003	**49.085**	**15.188**	**12.007**	-23.735	-27.380	-21.214	-14.140	-11.865	-6.900	-25.959	-28.110
2004-2008	**55.069**	**19.973**	**8.296**	-8.282	-7.378	-6.449	-0.211	**3.566**	**2.055**	-5.956	-18.687
2009-2013	**50.343**	**27.261**	**11.290**	-7.220	-9.734	-14.946	-5.463	-0.120	**6.684**	-9.493	-14.095
2014-2018	**53.780**	**34.773**	**20.671**	-13.643	-7.976	-7.995	-11.051	**10.862**	**13.279**	-14.366	-32.988

Notes on Table 6: The table shows the z-statistics for the test that the mean-square beta of the factor is significantly above the natural-rate benchmark value. See the text for the definition of the natural rate benchmark value. Numbers in bold are significantly greater than zero.

Table 7: Stepwise Regression Using the Akaike Information Criterion

	MKTRF	SMB	HML	USDSDR	GBPSDR	JPYSDR	KRWSDR	Oil	Gold	Alum	Lumber
1989-1993	*	*	*	5	6	4	3	*	*	2	1
1994-1998	*	*	*	5	4	6	2	7	*	3	1
1999-2003	*	*	*	2	3	5	6	*	*	4	1
2004-2008	*	*	*	*	*	*	*	*	*	*	1
2009-2013	*	*	*	*	*	*	*	*	*	*	1
2014-2018	*	*	*	*	*	*	*	*	*	*	1

Notes on Table 7: The table shows the results of stepwise regression using the AIC benchmark to select factors with positive explanatory power. Asterisks denote factors which pass the AIC test for inclusion at all steps. The integers denote the step order at which other factors were dropped (1 denotes the factor dropped after the first step, and so on).