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The Implications of Tail Dependency Measures for Counterparty Credit Risk Pricing☆

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Abstract

This paper investigates the counterparty credit risk of interest rate swaps positions using the credit valuation adjustment (CVA) measure, and examines the potential dependency relationships between the probability of default (PD) and exposure at default (EAD). We empirically tested, using interest rate swaption implied market volatilities, three tail dependency models: a Basel III Committee independent model, a Gaussian copula dependent model, and a Wrong Way Risk (WWR) with copula dependency approach. The results show that the CVA underestimation when using a Gaussian copula for modelling the dependence of PD and EAD is about 51%–362% compared to using WWR, and the underestimation between using the standardised Basel independent model and using the Gaussian copula is about 527%–1609%, including the period of the 2007/2008 crisis. This has important implications for regulators, financial institutions, and credit risk managers when calculating counterparty risk.

Keywords: Credit risk, Counterparty Credit Risk, Credit Value Adjustment, Dependency of credit risk components, Pricing swaps.

JEL Classifications: G10; G13; G33.

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1. Introduction

Banking regulation has advanced in establishing both control procedures and capital requirements, with bringing different impacts on financial institutions. The financial crisis in 2007/2008 highlighted the need for improvements in both credit risk measurement and management. Counterparty credit risk has become the focus of attention of both managers and regulators, since many losses incurred during the 2007/2008 crisis resulted from events associated with the deteriorating credit quality of market participants (BCBS, 2011b). Although credit risk has already been the subject of numerous studies, an important element that the crisis has brought to the fore involves pricing models of loan portfolios that have positions in derivatives. In this context, not only can actual events of defaults cause substantial devaluations in financial institutions' positions, but the deterioration of credit quality, of both banks and clients, can also affect the value of portfolios exposed to counterparty credit risk.[1]

The *BCBS* defines the *CVA* as the difference between the value of a portfolio with derivatives considering, respectively, the absence and the presence of credit risk of the counterparty (BCBS, 2011a). Importantly, capital requirements under the *CVA* are linked to portfolios of derivatives, distinguishing them from the traditional credit capital requirements associated with *Expected Losses (EL)* and *Unexpected Losses (UL)*. In fact, according to BCBS (2006) and BCBS (2011b), typical credit portfolios – for example, loans and financing – are analysed in the guidelines of Basel II, while portfolios exposed to counterparty credit risk in derivative transactions are analysed in Basel III.

In this research, our contribution to the literature is twofold: First, we calculate the *CVA* of interest rate swaps'[2] positions before, during, and after the 2007/2008 financial crisis. We establish a general modelling of *CVA* based on the discussion in Cherubini (2013), which allows us to study the elements of the model proposed by Sorensen and Bollier (1994) to price swaps exposed to counterparty credit risk. Later, we test a *WWR* copula-based model applied to the calculation of the *CVA* of swaps.[3] Additionally, we calculate a *CVA* with Gaussian copula dependence for comparison purposes. Our study uses risk-neutral implied volatilities[4] extracted from the interest rate swaption market from the period of the 10 May 2005 to 1 August 2013 for the pricing of the three *CVA* dependence models. Although we did consider the credit quality of

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[1] In fact, a study by the Basel Committee on Banking Supervision (*BCBS*) on the 2007/2008 crisis indicated that approximately 2/3 of credit losses came from the marked to market devaluations of positions exposed to counterparty credit risk and only 1/3 of losses came from the actual default of counter parties (BCBS, 2009). As a result, banking regulation has been directing its efforts to improve capital requirements, more specifically, by measuring the credit valuation adjustment *CVA* (*Credit Valuation Adjustment*), which is a metric to estimate marking losses on the market due to the exposure to counterparty credit risk.

[2] According to the Bank for International Settlements (BIS), by the end of the second semester of 2015, interest rate swaps instruments accounted for 288,634 billion US dollars in notional amount from a total of 492,911 billion US dollars of the over-the-counter (*OTC*) global derivatives market, a share of 58.56% of the total *OTC* derivatives market.

[3] Further analysis on copulas properties for tail dependency can be founs in Hua (2017), Su and Furman (2017), and Su and Hua (2017)

[4] We consider risk-neutral implied interest rate swaption volatilities, as there exists no market price for counterparty credit risk and, as Stulz (2004) and Bolton and Oehmkez (2013) report, during times of distress financial institutions have misvalued OTC derivatives in favour of diminishing the collateral associated with the instruments held, which could mislead the study.

the counter parties, our research aims to analyse the impact of modelling the wrong way risk *WWR* in the underestimation of counterparty risk between two large institutions acting as dealer and customer. Finally, we conduct sensitivity analyses to allow comparison of our results between traditional simple models based on independent risk components – such as, for example, the one suggested by the Basel Committee – and the dependency model based on copulas.

Our second contribution is that in light of the independent *PD* and *EAD* modelling for the *CVA* calculation proposed by BCBS (2011b) in their standardised settings, we measured the magnitude of the underestimated *CVA* of interest rate swaps positions in comparison with two alternative approaches: (i) a Gaussian copula[5] *CVA* dependence model and (ii) a *WWR CVA* dependence model. We find that the underestimation of the *CVA* of interest rate swaps might reach up to a 1600% in a *WWR* situation.[6]

The counterparty credit risk literature is extensive. General approaches for assessing the counterparty credit risk of derivatives portfolios include Hull (1989), Duffie and Huang (1996), and Jarrow and Yu (2001). Bomfim (2003) was the first study to assess the effects of the financial crisis[7] over the counterparty credit risk in interest rate swaps. Bomfim (2003) found that there existed no difference between synthetic interest rate swap constructed with Eurodollar future prices by a no-arbitrage relationship over the real interest rate swaps prices, indicating that there was no signalling effect before the crisis by the futures market. Counterparty credit risk models and studies on credit default swaps (*CDS*)[8] include Pan and Singleton (2008), Berndt et al. (2010), Arora et al. (2012), and Bo and Capponi (2015). Pan and Singleton (2008) developed and tested a new model for *CDS* terms structure that revealed the fact that default and recovery information was included in the term structure. Berndt et al. (2010) derived a Markovian affine pricing model within the Heath et al. (1992) multifactor framework, and in the empirical test they found that their model incorporates more information than the aggregate information of *CDS* index tranches. Arora et al. (2012) found evidence of counterparty risk premiums are priced in the *CDS* market and Bo and Capponi (2015) derived a new closed-form expression for *CVA* valuation in presence of several defaults.

Although the *BCBS* has suggested standardised settings for the *CVA*, simplifying the analysis and applying an exposure correction factor that reflects the *spread* used to discount a debt security with the equivalent credit risk of the counterparty (BCBS, 2009), criticisms from practitioners and academics have induced substantial changes in the criteria. Thus, the Basel Committee established a formula for the *CVA*

[5] Furman et al. (2016) reviewed Gaussian copulas properties for tail dependence modelling, and suggested the us of other types of copulas with greater tail dependence, as in our case the *WWR* risk copulas.

[6] Our calculations reveal, in an additional empirical exercise with bank regulatory data of the interest rate swap positions, that on average the 1600% CVA underestimation, will be equivalent to 1279% of the Bank capital, considering bank regulatory data on interest rate swaps positions from 1986 to 2017.

[7] The main driver of the 2007/2008 financial crisis was the market and the funding liquidity that stemmed the cascade effect on counterparty defaults. Brunnermeier and Pedersen (2009) developed a model that associated market liquidity with the funding liquidity of an institution and Brunnermeier (2009) described the origin and the evolution of the crisis through these market liquidity effects.

[8] Other studies of counterparty credit risk with other instruments include Hull and White (2006) for credit debt obligations (*CDO*), Lo et al. (2013) for catastrophe equity put options, and Sakurai and Uchida (2014) for cross-currency swaps.

based on a more detailed exposure to counterparty credit, taking into account the specific elements associated with credit risk components: probability of default (*PD*), loss given default (*LGD*) and exposure at default (*EAD*) of portfolios that include derivatives.

However, despite the evolution of the model in relation to the initial proposal of an adjustment factor, several areas of attention still exist, especially concerning the assumption of independence between the components of credit risk. In the modelling established by the *BCBS* to calculate the *CVA*, credit risk components are considered independent. The dependency among these risk components may be relevant and, therefore, not considering possible relationships can lead to inadequate measurement of risk. More specifically, the existence of the *WWR* can affect risk assessment, leading to the underestimation of potential losses when using models that only take into account independence among the credit risk components.

One of the first studies to assess the dependency of credit variables in counterparty credit risk was Hull and White (2001). The Hull and White (2001) model considered a correlation between the different institutions in the valuation of the *CDS*. More recently, Jorion and Zhang (2009) discovered an explanation for the 'contagion' (default correlation) effect of creditors with a large exposure. Jorion and Zhang (2009) found that the announcement of the bankruptcy of borrowers was strongly correlated with the *CDS* spreads for the creditors. Lipton and Sepp (2009) calculated the *CVA* via the Merton (1976) structural model for *CDS*, modelling the dependence of two factors by a joint jump intensity process. Gregory (2009) applied an earlier version of the Cherubini (2013) model of maximum bound copula *WWR* in Cherubini and Luciano (2002) to derive a general factor bivariate dependency model; nevertheless, the study was only theoretical, with no empirical findings. Eckert et al. (2016) provide a dependency model for the joint distribution of *LGD*, *PD*, and *EAD*; however, their model is based on the CreditMetrics normal distribution risk model and they do not provide empirical evidence.[9]

The *WWR* is the term commonly used to denote an unfavourable dependence between exposure to credit risk and the credit quality of the counterparty (BCBS, 2011b). The existence of the *WWR* is particularly problematic because it can exacerbate losses, since the exposure to credit risk can be higher at precisely the time when the probability of default increases. Alternatively, the probability of default may increase as the amount owed by a counterparty increases.

The results suggest that by not incorporating the dependence between credit risk components, many models underestimate the potential loss due to the counterparty credit risk. The analysis also indicates that the *WWR* is relevant, therefore the adverse and joint fluctuations of risk components should be evaluated for both risk management and for pricing purposes. Moreover, since the calculation of the *CVA* depends on the modelling, for example, of the dependency relationships of credit risk components, the diffusion process of

[9]Recent counterparty credit risk mathematical models such as Brigo and Chourdakis (2009), Hull and White (2012), Brigo et al. (2013a), Brigo et al. (2014), and Hull and White (2014) consider effects from collateralization, netting rules, and rehypothecation additional to the *WWR*; their use could provide valuable empirical insights about multivariate dependency effects on credit.

market risk factors, and volatility of risk factors, counterparty credit risk assessments can also be subject to risk modelling. In this sense, the study also shows, for the specific case of the *CVA* of swaps, that the definition of the dynamics of the interest rates diffusion process also influences the results of counterparty credit risk analysis.

The remainder of this study is structured as follows: Section 2 presents the notation and definitions for measuring counterparty credit risk and the *CVA*. In Section 3, the *CVA* calculation of interest rate swaps with the three different dependency models is presented. Section 4 describes the seven different interest rate models used for measuring the *CVA*. Section 5 describes the data and the empirical calibration method applied. Section 6 presents the results of the estimated volatility and the *CVA*. Section 7 provides conclusions, with suggestions for further extensions of the work.

2. Modelling tail dependence in counterparty credit risk

Let us consider two counter parties denoted by B (the bank) and C (the client). In this paper, we will use a risk-neutral probability measure, typically associated with pricing procedures, in contrast to the measure of real probability, usually linked to risk measurement mechanisms (Brigo et al., 2013b; Cherubini, 2013).[10] Notice that B does not necessarily represent a bank, but this notation makes it easier to adapt the discussion to market conventions. The bank or dealer (B) provides a product to the customer (C), which has a certain financial need. Given that, in most cases, the dealers have higher risk management capabilities, we will establish that the analysis of counterparty credit risk is being conducted by counterparty B.

*2.1. Credit valuation adjustment (*CVA*) definition*

We define $\Pi(t,T)$ as the net cash flows from t to T of a portfolio of financial products that B traded with C, discounted to the present value, and taking into consideration that C is not exposed to the credit risk of B. In this formulation, t is the time of the valuation of the portfolio and T is the tenor of the further cash flow to be exchanged between the counter parties. In the case of a portfolio with derivatives, T is the maturity of the contract with the greater expiration date.

The value of this portfolio in terms of price at time t is the expected value of $\Pi(t,T)$, i.e. $NPV(t) = \mathbb{E}_t[\Pi(t,T)]$, where NPV is the net present value. One can therefore consider that the fair value, without credit risk, of this portfolio of trades for counterparty B is worth $NPV(t)$. Many derivatives' pricing models were developed from this premise of no credit risk of both counter parties.

From a practical standpoint, however, the assumption of portfolios without credit risk is unrealistic, especially in the context of more complex operations such as the case of derivatives' transactions. Thus,

[10] Further details on the suitability of a risk-neutral probability measure with respect to the real-world probability measure can be seen, for example, in Gregory (2012) and Brigo et al. (2013b). Once having established the probability space and the analysis of structure based on a risk-neutral world, let us now discuss the concept of the *CVA*, emphasising the unilateral model, in which only one of the parties incurs in credit risk, since the other party has a negligible credit risk.

although we assume that bank B has zero probability of going bankrupt, i.e., is default-free, client C can go bankrupt. This assumption of the asymmetry of credit risk, whilst unreasonable, was usual in models previous to the 2007/2008 crisis, since it was considered that, for instance, AAA dealers were attributed an extremely low probability of not meeting their obligations. In general, the risk that client C was exposed to the credit quality was considered negligible.

The bankruptcy of several high-reputation international banks during the subprime crisis showed that even banks with AAA ratings could have difficulty in paying debts to their counter parties.

For modelling purposes, we consider that, as discussed, only bank B is exposed to the credit risk of the counterparty. We will analyse the value of a portfolio P from the point of view of the bank, with a possible default of the counterparty C in instant τ_C. Initially, two scenarios can be analysed: (1) the default occurs in $\tau_C > T$ or (2) the default occurs in $\tau_C \leq T$.

In the first case (1), the portfolio cash flows would not be affected by the default that would occur only after the expiration of the contract with higher maturity and therefore the evaluation follows the structure of a no counterparty credit risk loan portfolio given by $\Pi(t, T)$.

In the second case (2), when the event of default occurs before maturity or at maturity T, the result of B is associated with the present value of the flows that occur until default $\Pi(t, \tau_C)$ and the amount to be paid or received due to the mark-to-market value of the portfolio. In this context, two situations should be considered:

1. If the portfolio value is negative, i.e., if $NPV(\tau_C) < 0$, then counterparty B is in a losing position and pays cash flows to counterparty C, even if C has defaulted and hasn't paid its obligations to creditors. The value to be paid by B corresponds to $NPV(\tau_C) < 0$. Note that one of the assumptions of the model implies that B is default-free and will entirely honour its obligation to the counterparty; and

2. If the portfolio value is positive, i.e., if $NPV(\tau_C) \geq 0$, counterparty B receives $RR_C NPV(\tau_C)$, where RR_C represents the recovery rate.

Combining the various scenarios of the net cash flows in relation to the default of C, then the value of the portfolio of B exposed to credit risk of C can be defined as:

$$\tilde{\Pi}(t, T) = \mathbb{1}_{(\tau_c > T)} \Pi(t, T) + \mathbb{1}_{(t < \tau_C \leq T)} \Big[\Pi(t, \tau_C) + D(t, \tau_C) \Big(\min(NPV(\tau_C), 0) + RR_C \max(NPV(\tau_C), 0) \Big) \Big]. \tag{1}$$

According to the Basel Committee, for the specific case of a derivative contract, the CVA is typically defined as the difference between the value of the product assuming that the counterparty does not have default risk and the value of the product subject to counterparty default risk (BCBS, 2011a).

Let us now examine the definition of the *CVA*, following Brigo et al. (2013b). Let $\Pi(t,T)$ and $\tilde{\Pi}(t,T)$ be the net present values of the cash flows, measured at time t, of a portfolio maturing at T negotiated with a counterparty without credit risk and with credit risk, respectively. The *CVA*, i.e., the value of the portfolio value adjustment, depending on the counterparty credit risk, is given by the expected value of the difference between $\Pi(t,T)$ and $\tilde{\Pi}(t,T)$:

$$CVA_C = \mathbb{E}_t[\Pi(t,T) - \tilde{\Pi}(t,T)]. \tag{2}$$

From the definition in (2), one can get the unilateral *CVA*.

Proposition 2.1. *The credit value adjustment when only one of counter parties has credit risk is given by:*

$$CVA_C^B = \mathbb{E}_t[\mathbb{1}_{(t<\tau_c\leq T)} \cdot LGD_C \cdot EAD_C], \tag{3}$$

where,

$LGD_C = 1 - RR_C$ *is the loss given default (*LGD*; and,*

$EAD_C = (NPV(\tau_C))^+$ *is the exposure to counterparty credit risk in the instant of default (*EAD*).*

Proof. See Appendix A1. □

If we consider that the counterparty credit exposure, the loss given default, and the instant of default are independent and establishing $\mu_{PD_C} = \mathbb{E}[\mathbb{1}_{(t<\tau_C\leq T)}]$, $\mu_{LGD_C} = \mathbb{E}_t[LDG_C]$ and $\mu_{EAD_C} = \mathbb{E}_t[EAD_C]$, then the adjustment due to credit risk can be rewritten, in a simple way, as the product of the mean values of *PD*, *LGD* and *EAD*:

$$CVA_C^B = \mu_{PD_C} \cdot \mu_{LDG_C} \cdot \mu_{EAD_C}. \tag{4}$$

In Basel II, although the primary concern is not with the pricing of counterparty credit risk, *PD*, *LGD*, and *EAD* are called credit risk components and are used to estimate losses in traditional credit portfolios composed of loans and financing operations. Those components are fundamental for the calculation of regulatory capital requirements – for example, expected losses (*EL*) and unexpected losses (*UL*). It is important to notice that, in Basel II, a similar formula to (4) is used to estimate the expected loss of traditional credit portfolios.

The definition of the *CVA* given by (3), although it is more complex, since it involves portfolios of products that may have different characteristics, follows the logic of expected loss. In this context, the *CVA* is an estimate of expected losses calculated from the probability of the counterparty defaulting, the actual loss in the case of default, assuming that there is a certain recovery rate, and the value that an agent has to receive from the counterparty which defaulted.

2.2. Credit and equity risk tail dependence

WWR in a bivariate factor setting is related to the lower-lower tail risk of two factors being greater than the initial estimates. Nevertheless, we are interested in other multivariate dependence studies, for example, the dependence between credit and equity markets. For this purpose we use a Markov two-regime switching model (Hamilton, 1989) to assess the dependence between *PD*, *EAD*, and the stock market.

Vassalou and Xing (2004) used the Merton (1974) structural model to find evidence of the relationship between equity returns and default risks; their results suggest that a separation of the dataset into a two-regime equity could be important for the calculation of the *CVA*. Ang and Chen (2002) and Longin and Solnik (2001) found that there exist asymmetries between the bull and bear stock markets when the upper-upper (bull) and lower-lower (bear) tail returns correlations of stocks and the market are tested. Furthermore, Ang and Bekaert (2002) studied the impact of diversification in a Markov two-regime (bull vs. bear) setting of (i) all-equity portfolios and (ii) equity plus a conditional risk-free asset, finding that when higher lower-lower tail dependence is considered only portfolios with the conditional risk-free asset have a significant difference, as compared to those portfolios that dismiss the existence of a two-regime setting.[11]

In our study we will divide the data into two regimes: (i) a regime considered to be 'normal' with average mean-volatilities, (ii) a regime considered to be of 'crisis' with different mean to model asymmetry and higher volatility. Let y_t be the stock univariate process, $x_{i,t}$ be the explanatory variables, $x_{i,t}^{nS}$ be the subset of $x_{i,t}$ that has non-switching variables, $x_{i,t}^{nS}$ be the subset of $x_{i,t}$ with the switching variables, $S_t \in \{1, \dots, k\}$ be a variable that represents the state at time t, and β_i, ϕ_{j,S_t} be the parameters; a Markov regime switching process can be defined as:

$$y_t = \sum_{i=1}^{N_{nS}} \beta_i x_{i,t}^{nS} + \sum_{j=1}^{N_s} \phi_{j,S_t} x_{j,t}^{S} + \epsilon_t, \quad (5)$$

$$\epsilon_t \sim P(\Phi_{S_t}), \quad (6)$$

where $P(\Phi_{S_t})$ is the distribution assumed for the innovations. Consider a two-regime model with differences in mean and variance between the regimes, with Gaussian error innovations, then $k = 2$ and (5) reduce to:

$$y_t = \beta_{i,S_t} x_{1,t} + \beta_{2,S_t} x_{2,t} + \epsilon_t, \quad (7)$$

$$\epsilon_t \sim N(0, \sigma_{S_t}^2). \quad (8)$$

To implement the Markov two-regime switching model we used the *MATLAB MS_ Regress* toolbox from Perlin (2010). Once the data has been divided into the two regimes, we tested the *PD* and the EAD dependence on each regime.

[11]Our time-varying tail dependence analysis can be used jointly with Boonen et al. (2017) model on non-linear risks aggregation to improve the analysis in capital allocation exercises.

3. Pricing of swaps exposed to counterparty credit risk

As already discussed, the *CVA* in (3) includes the model from Basel III as a special case. The *CVA* in (3) also incorporates, as a particular case, the seminal model of Sorensen and Bollier (1994), which analysed the pricing of swaps subject to counterparty credit risk. Despite the fact that the *CVA* can be applied to portfolios of financial products traded with a certain counterparty, the modelling of individual transactions is relevant as it allows us to focus on the pricing of specific operations, without any influence from other products in the portfolio.

For example, derivatives' portfolios commonly have trades that may have cash flows in opposite directions, market to market from netted positions. Thus, an interest rate derivative may have cash flows that match those of a commodity derivative, traded with the same counterparty in the over-the-counter (OTC) market. From the point of view of counterparty credit risk analysis, the cash flows from these trades can be grouped together. In addition, various market risk factors can be interrelated, implying a diversification effect, which in turn can mitigate some of the counterparty credit risk. However, from the perspective of analysis of the derivative itself, each product can be viewed individually. Thus, the analysis of specific products is also relevant for pricing derivatives' products subject to the credit risk of the counterparty, as the sum of the *CVA*s of the portfolio may differ from the sum of the *CVA*s of the derivatives that comprise the portfolio.

3.1. Swap analysis

We will focus our analysis of the *CVA* on traditional swaps. We will initially consider bank B bank holding a long/short position in the swap. As discussed in Section 2, we will establish, for example, that bank B does not impose credit risk on its counterparty C. However, client C, i.e., the counterparty of bank B, may go bankrupt.

Without loss of generality, the notional value is defined as 1 monetary unit. The net present value $NPV^B(t)$, measured in t, associated with bank B's cash flows is given by:

$$NPV^B(t) = \sum_{i=1}^{n} D(t,t_i)\delta_i(f(t,t_{i-1},t_i) - r_s(t_0,t_n)), \tag{9}$$

where:

$D(t,t_i)$ is the discount factor of cash flows in a risk-neutral world;

$\delta_i = t_i - t_{i-1}$ is the difference between the tenor of two subsequent cash flows;

$r_s(t_0,t_n)$, denoted by the swap rate, is the spot rate that will be applied to the fixed leg of the swap; and

$f(t,t_{i-1},t_i)$ is the forward rate, established in t, related to the spot rate that will be be applied to cash flows in t_{i-1} with maturity in t_i.

Note that the swap rates r_s and the forward rates f, as already indicated, are given in the same unit of time horizons as t. At the time of the trade t_0, the swap should reflect a balance between the expectations of accrued interest through a fixed rate r_s and forward rates f until the maturity of the operation. Thus, the swap $r_rates(t_0, t_n)$ is the fixed rate that corrects the notional value each period, remaining constant between the whole interval from the moment of the operation t_0 until maturity t_n. The forward rates $f(t_0, t_{i-1}, t_i)$ represent the interest rates that the market, at instant t_0, considers appropriate to be in place from t_{i-1} for the period δ_i, taking into account the terms of the swap.

To prevent arbitrage, the cash inflows and outflows for a given counterparty, at instant t_0, must be equal and therefore $NPV^B(t_0) = 0$:

$$\sum_{i=1}^{n} D(t_0, t_i)\delta_i(f(t_0, t_{i-1}, t_i)) = \sum_{i=1}^{n} D(t_0, t_i)\delta_i(r_s(t_0, t_n)). \tag{10}$$

Since $r_s(t_0, t_n)$ is fixed for all periods of the swap then, from a non-arbitrage perspective, the fixed leg will be adjusted by a rate, defined in the instant of the trade, equivalent to:

$$r_s(t_0, t_n) = \frac{\sum_{i=1}^{n} D(t_0, t_i)\delta_i(f(t_0, t_{i-1}, t_i))}{\sum_{i=1}^{n} D(t_0, t_i)\delta_i}. \tag{11}$$

From the perspective of risk analysis, if $NPV^B(t) < 0$, the investor incurs losses as a result of market risk, but is not exposed to credit risk, since there is no money to receive from the counterparty. In contrast, if $NPV^B(t) > 0$, the investor is in a winning position in terms of market risk, but with an exposure to credit risk, considering that he may receive the entitled flows if the counterparty defaults.

In this context, exposure to credit risk at any instant $t > t_0$ before maturity T is given by $\max(NPV^B(t), 0)$, which would involve the need to compare $r_s(t_0)$ with the new rates to term $f(t, t_{i-1}, t_i)$ concerning the time periods for the cash flows to win between t and T. However, a more pragmatic exposure analysis engine involves comparing the fixed swap rate at the time of the trade $r_s(t_0, t_n)$ with a fixed interest rate swap at time t that is, $r_s(t, t_n)$, which reflects forward rates $f(t, t_i, t_j)$ in t, for maturities t_i, $i = 1,\ cdots, n-1$. This new rate $r_s(t, t_n)$ refers to the equilibrium value of a new swap transaction at time t.

If, due to default, the contract expires in τ_C, with $t_{j-1} \leq \tau_C \leq t_j$, then the exposure to credit risk is given by the positive present value that occurs when $sr(\tau_C, t_n) > sr(t_0, t_n)$, in the case of bank B paying the fixed rate and receiving the floating rate. Under these conditions, the future exposure of bank B, if the counterparty defaults, is equivalent to the present value:

$$NPV^B(\tau_C) = \sum_{i=j}^{n} D(\tau_C, t_i)\delta_i \max[r_s(\tau_C, t_n) - r_s(t_0, t_n), 0]. \tag{12}$$

If the counterparty defaults, all future receivable or payable cash flows are discounted to the present value at the instant of default τ_C and the transaction is settled. The amount must be paid by the debtor counterparty to the winning counterparty. Considering a unilateral exposure, in which only counterparty B is exposed to credit risk, then the amount at risk involves the positive value of the notional amount of the difference adjusted by the swap rates in τ_C and t_0.

The exposure shows an analogy with an option to enter into a swap, paying rate $r_s(t_0, t_n)$ and getting rate $r_s(\tau_C, t_n)$. This result is equivalent to holding a long position in an option to enter, at a later date, $t = \tau_C$, in a swap paying a strike rate $r_s(t_0, t_n)$. In this case, the lender has an option to enter into a swap in the long position. This option to choose between getting or not into a swap at a later date is called swaption.

For the formula of Cherubini (2013), we consider (3) and follow some assumptions consistent with the guidelines of Basel III. Let us establish that the LGD, and hence the recovery rate, is independent of time of and of the credit exposure level. Despite being used in many studies, this assumption may be unrealistic, as shown by the work of Altman (2001) and Folpmers (2012). However, the focus of this analysis involves the study of the relationship between the event of default and the exposure at the instant of default:

$$CVA_C^B = LGD_C \cdot \mathbb{E}_t\Big[\mathbb{1}_{(t<\tau_c\leq T)} \cdot EAD_C\Big], \tag{13}$$

where the superscript index shows the counterparty whose position is adjusted for the credit risk of the counterparty that, in turn, is denoted in the subscript index.

To calculate the CVA from (3), we will analyse the exposure to credit risk of the counterparty jointly with the default. As previously stated, if the counterparty C defaults in $t_{j-1} \leq \tau_C \leq t_j$, the contract is settled and counterparty B is exposed to the credit risk if it has a profitable position. Thus, simultaneously, two conditions must be met: (1) the counterparty C should enter into default in a moment $t \leq \tau_C \leq t_n$ and (2) if bank B is long in the swap, its exposure should be positive, i.e., $\max[r_s(\tau_C, t_n) - r_s(t_0, t_n), 0] > 0$.

Note that, when these conditions are met, the present value of the exposure given by (13) is $NPV^B(\tau_C) \geq 0$. Therefore, considering (3), the likelihood that the default occurs at any time t and there will be n cash flows, the present value in τ_C of the remaining cash flows, without taking into account the changes in the rates of the fixed and floating swap, is given by $A(\tau_C, t_j, t_n) = \sum_{i=j}^{n} D(\tau_C, t_i)$. When there is a default, the exposure is given by:

$$EAD_C = A(t_C, t_j, t_n) \cdot \mathbb{E}_t\left((\max[r_s(\tau_C, t_n) - r_s(t_0, t_n), 0])\right). \tag{14}$$

The expectation can be obtained by identifying the potential positive differences between the swap rates, i.e., in the interval that $r_s(\tau_C, t_n) \geq r_s(t_0)$, when default occurs in $t_{j-1} \leq \tau_C \leq t_j$. Therefore:

$$\mathbb{E}_t\left((\max[r_s(\tau_C, t_n) - r_s(t_0, t_n), 0])\right) = \int_{sr(t_0,t_n)}^{\infty} \mathbb{P}(r_s(t_j, t_n) > u, t_{j-1} \leq \tau_C \leq t_j)\, \mathrm{d}\, u. \tag{15}$$

In this context, the CVA can be calculated as

$$CVA_C^B(t) = LGD_C \cdot \sum_{j=1}^{n} A(t,t_j,t_n) \cdot \int_{sr(t_0,t_n)}^{\infty} \mathbb{P}(r_s(t_j,t_n) > u, t_{j-1} \leq \tau_C \leq t_j)\,\mathrm{d}\,u. \tag{16}$$

Therefore, the CVA for bank B, with a long position in the swap, and exposed by counterparty credit risk of C, is given by:[12,13]

$$CVA_C^B(t) = LGD_C \cdot \sum_{j=1}^{n} A(t,t_j,t_n) \cdot \int_{r_s(t_0,t_n)}^{\infty} \tilde{\mathbb{C}}(1-\mathbb{Q}(u), \mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j))\,\mathrm{d}\,u. \tag{17}$$

3.2. Analysis of dependence

The model in (17), defined using copulas, can be used to analyse several relationships between the probability of default defined by $(\mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j))$ and the exposure at default defined by $\int_{r_s(t_0,t_n)}^{\infty}(1 - \mathbb{Q}(u))\,\mathrm{d}\,u$. Considering initially the instant of default independent of the exposure at default, then the copula function to be used is $\tilde{\mathbb{C}}(u,v) = uv$. Since $\int_{r_s(t_0,t_n)}^{\infty} \tilde{\mathbb{Q}}(u)\,\mathrm{d}\,u = \mathbb{E}_Q[\max(r_s(t_j,t_n) - r_s(t_0,t_n), 0]$, with $\tilde{\mathbb{Q}} = 1 - \mathbb{Q}$, the expectation represents the price of an option on a swap, since the counterparty would have the right to get into a swap in favourable market conditions. Then the adjustment related to counterparty credit risk is given by:

$$CVA_C^B(t) = LGD_C \cdot \sum_{j=1}^{n} (\mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j))\mathrm{SwOp}(t, t_j, r_s(t_0,t_n)), \tag{18}$$

where $\mathrm{SwOp}(t, t_j, r_s(t_0,t_n))$ is the price in t of an option on a swap, in which bank B has the right to go long in a swap at t_j, paying a fixed rate $r_s(t_0,t_n)$.

As noted by Cherubini (2013), this result is analogous to the work of Sorensen and Bollier (1994), which is one of the first and most influential papers on the counterparty credit risk of derivatives. According to Sorensen and Bollier (1994), the exposure of counterparty B to the credit risk of counterparty C in a swap contract is equivalent to a series of European style options on swaps that are exercised by B to get reimbursed

[12] The model in (17) also incorporates the Basel guidelines. Cherubini (2013) shows that if we define $EE_j = \frac{A(t,t_j,t_n)}{D(t,t_j)} \mathbb{E}_Q[\max(r_s(t_j,t_n) - r_s(t_0,t_n), 0]$, then, $CVA_C^B(t) = LGD_C \cdot \sum_{j=1}^{n} (\mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j)) D(t,t_j) EE_j$. Assuming a discretisation that considers the exposure as an average between two periods, $(EE_{j-1} + EE_j)/2$, one can obtain the counterparty credit risk indicated by the Basel Committee. Thus, the model from Cherubini (2013) also includes other simpler definitions of CVA.

[13] The important contribution from Cherubini (2013) for the calculation of the CVA involves representation of a joint probability using copulas, which allows an analysis of many other derivatives' products and the study of the relationships among credit risk components. Following Cherubini (2013), $\mathbb{Q}(u) = \mathbb{P}(r_s(t_j,t_n) \leq u)$ is the risk-neutral probability of the swap rate being less than a given value u and considering an average default intensity for each term t_j given by $\lambda_i = s_i/LGD$, with survival probability t_i defined by $S(t_i) = \exp(-\lambda_i t_i)$, then: $\mathbb{P}(r_s(t_j,t_n) > u, t_{j-1} \leq \tau_C \leq t_j) = \tilde{\mathbb{C}}(1 - \mathbb{Q}(u), \mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j))$, where: $\tilde{\mathbb{C}}(1-z, v)$ is the copula function associated with the joint event of the swap rate in a given instant t being higher than the swap rate in the trade date t_0 and the counterparty C getting into default in an instant between t_{j-1} and t_j (Cherubini, 2013).

from losses due to the default of C. It is important to emphasise that the model from Sorensen and Bollier (1994) works under conditions of independence among the credit risk components and aggregates LGD and EAD in a single element.

Assuming now that there is dependence among the components of counterparty credit risk, more particularly, between PD and DL, considering that exposure may depend on the time of default that, in turn, derives from the survival probability of a counterparty in a given time interval. Analysing the WWR from the perspective of counterparty B holding a long position in the swap, the exposure to credit risk increases when the value of the swap increases, i.e., the greater $r_s(t, t_n)$ compared to $r_s(t_0, t_n)$, the higher the value of the position for the bank, but the higher the credit risk to which the bank is exposed.

A situation of WWR involves, for the buyer of the swap, a scenario in which the interest rate increases, implying greater exposure to credit risk and, at the same time, a higher likelihood of default, given by the difference between the survival probability of C between the two periods. Considering an extreme case, defined by the copula $\tilde{\mathbb{C}}(u, v) = \min(u, v)$, following Cherubini (2013):

$$\tilde{\mathbb{C}}(1 - \mathbb{Q}(u), \mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j)) = \min(1 - \mathbb{Q}(u), \mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j)) \tag{19}$$

Proposition 3.1. *Consider a perfect dependence between the exposure and the default of the counterparty. In the worst case scenario (*WWR*) the* CVA *of a long position in a interest rate swap is:*

$$CVA_C^B = LGD_C \cdot \sum_{j=1}^{n} A(t, t_j, t_n) \times \left(\max(k(t_j) - r_s(t_0, t_n), 0)(\mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j)) + \mathbf{SwOp}_P(t, t_j, \max(r_s(t_0, t_n), k(t_j))) \right), \tag{20}$$

where:

$\mathbf{SwOp}_P(\cdot)$ *represents the value of a payer swaption, in which one has the right, but not the obligation, to get into a swap paying a fixed rate and receiving a floating rate,*

and the worst case scenario CVA *of a a short position in a interest rate swap is:*

$$CVA_B^C = LGD_C \cdot \sum_{j=1}^{n} A(t, t_j, t_n) \times \left(\min(r_s(t_0, t_n) - k^*(t_j), 0)(\mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j)) + \mathbf{SwOp}_R(t, t_j, \min(r_s(t_0, t_n), k^*(t_j))) \right), \tag{21}$$

where:

$\mathbf{SwOp}_R(\cdot)$ *represents the price of a receiver swaption, in which the investor has the option to get into a* *swap paying a floating rate and receiving a fixed rate.*

Proof. See Appendix A2. □

Thus, through the use of extreme copulas, the model of Cherubini (2013) enables the study of a swap price considering the credit risk of the counterparty, including situations of dependence or independence between the probability of default and exposure at default.

An additional dependency model is tested. We replaced the copula in (19) by a Gaussian copula with the purpose of testing different dependency models:

$$\tilde{\mathbb{C}}_G(1-\mathbb{Q}(u), \mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j)) = \Phi\left(\Phi\left(1-\mathbb{Q}(u)\right)^{-1}, \Phi^{-1}\left(\mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j)\right)\right) \tag{22}$$

Using the Proposition 3.1, it is straightforward to approximate the CVA of a long position interest rate swap position with a Gaussian copula for the dependence between the EAD and the PD by:

$$CVA_C^B = LGD_C \cdot \sum_{j=1}^{n} A(t, t_j, t_n) \cdot \Phi\left(\Phi^{-1}\left(\mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j)\right), \Phi^{-1}\left(\mathbf{SwOp}_P(t, t_j, r_s(t_0, t_n))\right)\right), \tag{23}$$

and CVA for a short position in the interest rate swap by:

$$CVA_B^C = LGD_C \cdot \sum_{j=1}^{n} A(t, t_j, t_n) \cdot \Phi\left(\Phi^{-1}\left(\mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j)\right), \Phi^{-1}\left(\mathbf{SwOp}_R(t, t_j, r_s(t_0, t_n))\right)\right), \tag{24}$$

3.3. Pricing of swaptions

In order to apply the model from Cherubini (2013) for swaption pricing, we will briefly describe valuation methods for bonds, swaps, and options on swaps, using interest rates diffusion processes.

Let $r(t)$ be an interest rate diffusion process. We consider a general formula for the pricing of a zero coupon bond, with unity face value at maturity. Then the price of the zero coupon bond is:

$$P(t,T) = \mathbb{E}_t\left[\exp\left(-\int_t^T r(s)ds\right)\right]. \tag{25}$$

Following Duffie et al. (2000), (25) can be written as:

$$P(t,T) = A_1(t,T)\exp(-B_1(t,T)r(t)), \tag{26}$$

where:

$A_1(t,T) = \frac{P^M(0,T)}{P^M(0,t)}\exp[B_1(t,T)f^M(0,t) - \frac{\sigma^2}{4a}(1-\exp(-2at))B_1(t,T)^2]$;

$B_1(t,T) = \frac{1}{a}[1-\exp(-a(T-t))]$;

$P^M(0,x)$ is the discount factor for a cashflow in x, obtained from market data; and

$f^M(0,x)$ is the instantaneous interest rate observed in the market in 0 with expiration in T.

Considering (20) and (21), the *CVA* depends on the price of swaptions. Then the payer swaption and the receiver swaption prices are given, respectively, by:

$$\mathrm{SwOp}_P(t,T,X) = \sum_{i=1}^{n} c_i \mathrm{Put}(t,T,t_i,X_i), \tag{27}$$

$$\mathrm{SwOp}_R(t,T,X) = \sum_{i=1}^{n} c_i \mathrm{Call}(t,T,t_i,X_i), \tag{28}$$

where:

$c_i = X\delta_i$, for $i = 1, \cdots, n$ with $\delta_i = t_i - t_{i-1}$.

The values in t of European Call and Put options with strike price X, and maturity T, associated with a zero-coupon bond as the underlying asset which expires in S, are given, respectively by:

$$\mathrm{Call}(t,T,S,X) = P(t,S)\Phi(h) - XP(t,T)\Phi(h-\sigma_p),$$

$$\mathrm{Put}(t,T,S,X) = XP(t,T)\Phi(-h+\sigma_p) - P(t,S)\Phi(-h),$$

where:

$\sigma_p = \sigma\sqrt{\frac{1-\exp(-2a(T-t))}{2a}}B(T,S)$,

$h = \frac{1}{\sigma_p}\ln(\frac{P(t,S)}{P(t,T)X}) + \frac{\sigma_p}{2}$.

To calibrate parameters and price these options, we also need data of derivatives on swaps. Regarding the swap pricing procedure, we use a local volatility model, in which parameters $k(t_j)$ and $k^*(t_j)$ are calculated as follows (Cherubini, 2013):

$$k(t_j) = \mathrm{fsr}(t,t_j,t_n)\cdot\exp\left[-\frac{\sigma_{j,n}}{2} - \Phi^{-1}(\mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j))\sigma_{j,n}\sqrt{t_j - t}\right], \tag{29}$$

$$k^*(t_j) = \mathrm{fsr}(t,t_j,t_n)\cdot\exp\left[-\frac{\sigma_{j,n}}{2} + \Phi^{-1}\left(\mathbb{S}_C\left(t_{j-1}\right) - \mathbb{S}_C\left(t_j\right)\right)\sigma_{j,n}\sqrt{t_j - t}\right], \tag{30}$$

where:

$\mathrm{fsr}(t,t_j,t_n)$ is the forward swap rate, i.e., the rate defined in instant t for a swap beginning in t_j and expiration in t_n;

$\sigma_{j,n}$ is the volatility associated with a trade with a similar tenor; and

$\Phi(x)$ represents a standard cumulative normal distribution.

4. Description of interest rate models

The counterparty credit risk in interest rate swap transactions depends on the diffusion process of the relevant risk factors. Thus, the assessment of the *CVA* is influenced not only by choosing the form of dependence between the risk components (*PD*, *LGD*, and *EAD*) but also by the model used to express the dynamics of interest rates. According to Buetow et al. (2001), there are two distinct approaches to the structuring of stochastic differential equations to model the behaviour of interest rates: (a) equilibrium models and (b) no-arbitrage models.

The equilibrium models seek to establish mechanisms of pricing of debt securities under an analytical framework based on market equilibrium, which specifies the market price of risk (Buetow et al., 2001; Vasicek, 2007). Some equilibrium models are traditionally studied, such as Vasicek (1977), Brennan and Schwartz (1979), Cox et al. (1985), Longstaff (1989), Longstaff (1992), and Dai and Singleton (2003).

In contrast, no-arbitrage models use market prices to generate a grid of possible values of interest rates, by which the theoretical price of a debt security is equal to the price observed in the market. Traditional models of non-arbitration are Ho and Lee (1986), Black et al. (1990), Hull and White (1990), Black and Karasinski (1991), Heath et al. (1992), and Hull and White (1993). Importantly, no-arbitrage models allow parameters to be calibrated according to the prices traded in the market, and therefore offer greater comparability to values traded by practitioners.

The *CVA* calculated in (17) establishes a model in which the instant of default, associated with a survival function and therefore with the probability of default (*PD*), can be studied jointly with the exposure at default (*EAD*).

We now calculate the *CVA* of an interest rate swap, using interest rates and volatilities' market data. We focus on the analysis of no-arbitrage interest rate models. To model the diffusion process of interest rate r, we will use three one-factor short-term interest rate models: Black et al. (1990), Hull and White (1990), and Black and Karasinski (1991), the multifactor interest rate model of Heath et al. (1992), the two-additive-factor interest rate model of Brigo and Mercurio (2003) known as *G2++*, the stochastic volatility model of Hagan et al. (2002) better known as *SABR*, and the *LIBOR* market model of Brace et al. (1997).

4.1. One-factor short-term interest rate models

The model from Black et al. (1990), although initially defined for a discrete analysis, can be adjusted to incorporate a continuous stochastic differential equation:

$$\mathrm{d}\ln(r(t)) = \Theta(t)\,\mathrm{d}\,t + \sigma(t)\,\mathrm{d}\,W(t), \tag{31}$$

where $r(t)$ is the instantaneous interest rate, $\Theta(t)$ is the mean reverting equilibrium, $\sigma(t)$ is the instantaneous short rate volatility, and $W(t)$ is a Brownian motion.

Hull and White (1990) and Hull and White (1994b) present an extension of the Vasicek (1977) model, in which interest rate dynamics are given by:

$$\mathrm{d}\, r(t) = [\vartheta(t) - \alpha(t) r(t)]\, \mathrm{d}\, t + \sigma(t)\, \mathrm{d}\, W(t), \tag{32}$$

where $\vartheta(t)$ is the mean reverting equilibrium, and $\alpha(t)$ is the rate on which $r(t)$ reverts to $\vartheta(t)$.

Black and Karasinski (1991) established that the dynamics of the spot rates follow a generalised model from Black et al. (1990):

$$\mathrm{d} \ln(r(t)) = [\theta(t) - a(t) \ln(r(t))]\, \mathrm{d}\, t + \sigma(t)\, \mathrm{d}\, W(t), \tag{33}$$

where $\theta(t)$ is the mean reverting equilibrium, $a(t)$ is the rate on which $\ln(r(t))$ reverts to $\theta(t)$. Another expression for this model is:

$$d \ln(r) = a(t) [\ln(\mu(t)) - \ln(r(t))]\, dt + \sigma(t)\, \mathrm{d}\, W(t), \tag{34}$$

where $\ln(\mu(t))$ is the mean reverting level; if $\log(r) > \log(\mu(t))$ is superior then the interest rate will decrease, and if $\log(r) < \log(\mu(t))$ then the interest rate will increase. The resulting interest rate will have a lognormal distribution, that is the continuous time limit of the Black et al. (1990) model. Binomial and trinomial trees will be used to price the short rate models.

The model from Hull and White (1990) can be calibrated to the term structure of interest rates and to the spot or forward term structure of volatility. However, a perfect adjustment to the term structure of interest rates may imply problems in the adjustment to the volatility term structure, since not all volatilities extracted from market prices are relevant, due to the lack of liquidity of financial products and to the fact that the future volatilities' term structure in (32) may not follow a realistic traditional shape (Carverhill, 1995; Hull and White, 1995; Brigo and Mercurio, 2006).

4.2. Heath, Jarrow, and Morton (1992) model

Taking into account the general model, given by (32), Heath et al. (1992) developed an analysis that follows Hull and White (1990) and Hull and White (1994a), using an extension of the Vasicek (1977) model in which a and σ are positive constants and ϑ can be calibrated to adjust to the implicit market interest rates.

In contrast with the previous models, the interest rate dynamics in Heath et al. (1992) follow the diffusion process, given a maturity T:

$$\mathrm{d}\, f(t,T) = \alpha(t,T)\, \mathrm{d}\, t + \sigma(t,T)\, \mathrm{d}\, W(t), \tag{35}$$

with $f(0,T) = f_M(0,T)$ representing the term structure of interest rates given by the market price in $t = 0$, where:

$W = (W_1, \cdots, W_N)$ follows a Brownian motion of N dimensions;

$\sigma(t,T) = (\sigma_1(t,T), \cdots, \sigma_N(t,T))$ is a vector of adaptive processes related to volatilities; and

$\alpha(t,T)$ is an adaptive process, chosen from the vector of volatilities σ and the rate of evolution of the dynamic of the N zero coupon bonds selected to the pricing model.

Instead of modelling the short-term interest rate, Heath et al. (1992) use the whole interest rate term structure information by modelling the instantaneous forward rate curve. The forward rate curve includes information of the future rates that eventually will become the short rate at some point in time.

*4.3. Linear Gaussian two-additive-factor (*G2++*) model*

Given the widespread use of the Hull and White (1993) model, Brigo and Mercurio (2003) developed an extension with two linear Gaussian factors to overcome certain smile modelling limitations without losing the simplicity of the model. Let $x_g(t), y_g(t)$ be two Gaussian factors, the interest rate evolution is denoted by:

$$\begin{aligned} r(t) &= x_g(t) + y_g(t) + \phi(t), \\ \mathrm{d}\, x_g(t) &= -a(t) x_g(t)\,\mathrm{d}\, t + \sigma(t)\,\mathrm{d}\, W_1(t), \\ \mathrm{d}\, y_g(t) &= -b(t) y_g(t)\,\mathrm{d}\, t + \eta(t)\,\mathrm{d}\, W_2(t), \end{aligned} \tag{36}$$

with $x_g(0) = y_g(0) = 0$, where $\sigma(t)$ and $\eta(t)$ are the corresponding volatilities associated with the factors $x_g(t), y_g(t)$, respectively. The *G2++* model is highly tractable and we will provide some of the swaption's calculations for measuring counterparty credit risk *CVA* using a *G2++* analytical formula. We also provide swaptions calculations using a numerical simulation method.

4.4. LIBOR *market model*

Interest rate market models were created to match the prices observed in some interest rate derivatives such as caps and swaptions. These derivatives are priced by practitioners with the Black and Scholes (1973); Black (1976) formulae, as in the Black and Karasinski (1991) model. The forward-*LIBOR* (London Interbank Offered Rate) market model, or just the *LIBOR* model, was developed by Brace et al. (1997) as an extension of Heath et al. (1992) when modelling simple but observable M forward rates $f_i(t), i \in \{1, \ldots, M\}$. Let $f_i(t)$ represent the multivariate Gaussian process of the forward rates between t_{i-1} and t_i, $\sigma_i(t)$ the instantaneous volatility of $f_i(t)$, $\tau_i = t_i - t_{i-1}$, and $\mathrm{d}\, W_i(t)\,\mathrm{d}\, W_j(t) = \rho_{i,j}$, then the Gaussian *LIBOR* market model is defined by:

$$\frac{\mathrm{d}\, f_i(t)}{f_i(t)} = \xi\sigma_i(t)\,\mathrm{d}\sum_{k=I_B}^{I_E}\frac{\rho_{i,k}\tau_k\sigma_k(t)f_k(t)}{1+\tau_k f_k(t)}\,\mathrm{d}\,t+\sigma_i(t)\,\mathrm{d}\,W_i(t), \tag{37}$$

where

$$\begin{aligned} &I_B = i+1, I_E = j, \xi = 1, && i<j,\\ &\xi = 0, && i=j,\\ &I_B = j+1, I_E = i, \xi = -1, && i>j. \end{aligned}$$

LIBOR market models are priced using a numerical simulation.

*4.5. Stochastic alpha, beta, rho (*SABR*) market model*

The *SABR* model developed by Hagan et al. (2002) is an extension of the market models where the forward rates are modelled using a modified constant elasticity of variance (*CEV*) diffusion with stochastic volatility. Three parameters define this model: α represents the vol-vol or volatility of the volatility, β represents a leverage effect of increasing volatility with falling prices (and vice versa), and ρ the correlation between the forward rate diffusion and the volatility diffusion; the *SABR* model can then be defined by:

$$\begin{aligned} \frac{\mathrm{d}\, f_i(t)}{f_i(t)^\beta} &= \sigma(t)\,\mathrm{d}\,W_i(t), \\ \mathrm{d}\,\sigma(t) &= \alpha\sigma(t)\,\mathrm{d}\,Z_i(t), \\ \mathrm{d}\,W(t)\,\mathrm{d}\,Z(t) &= \rho\,\mathrm{d}\,t. \end{aligned} \tag{38}$$

Numerical simulations will be used to price swaptions under the *LIBOR* model.

5. Data description and estimation procedure

For the research we used different datasets. The first dataset is used for calculating the zero-coupon interest rate term structure.

5.1. Zero-coupon interest rate term structure estimation

To determine the implied zero coupon rates we use daily closing prices data from the US *LIBOR* from overnight up to 6 months for the short-term part of the curve, and US interest rate swap markets for maturities that range from 1 year up to 40 years for the medium- to the long-term part of the term structure. *LIBOR* represents the average interest rate at which the banks in London will lend between them in American dollars. As the fixed-income markets are in general OTC, then we use the data provided by the Bloomberg platform. A detailed description of the instruments is in Table 1.

****
Please insert Table 1 here.
****

The data span from 10 May 2005 to 29 January 2015 (Figure 1). Maturities for the selected *LIBOR* instruments range from spot (overnight rate) to 6 months. Swap rates are provided as the implicit rates from the interest rate swap between two counter parties on which the payer will have to pay a fixed rate and will receive the spot *LIBOR* rate. Bloomberg swap rates are calculated from the Treasury bonds' mid-prices and the quoted swap spreads.

****
Please insert Figure 1 here.
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Discount factors and forward rates are estimated from the zero-coupon rates (swap rates). With discount factors and forward rates, the fixed rate of the interest rate swap (the swaption strike price) can be calculated. Similarly, forward swap rates (*FSR*) with an specific tenor (2, 5, and 10) are calculated using discount factors and forward rates. Finally, the interest rate term structure is estimated using a Nelson and Siegel (1987) curve fitting, that will be useful for extrapolating the rates of intermediate maturities (Figure 2).

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Please insert Figure 2 here.
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5.2. Implied risk-neutral volatilities estimation

The second dataset used in this research is the risk-neutral interest rate volatility, extracted from the caps/floors volatility (Hagan and Konikov, 2004). This risk-neutral market volatility defined by the Black (1976) volatility surface is used for the calibration of the different interest rate models utilised in pricing the *CVA*. The volatility is extracted for the caps/floors with strikes of 1%, 2%, 3%, 4%, 5%, 6%, 7%, 8%, 9%, 11%, 12%, 13%, and 14%, with maturities that range from 2 up to 10 years. Figure 4 shows the market caps/floors volatility for the 1% and 5% strikes. The data were collected for the period from 10 May 2005 to 1 August 2013. In Figure 3 we observe the volatility surface for 10 May 2005.[14]

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Please insert Figure 4 here.
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[14] Duyvesteyn and de Zwart (2015) calculate the volatility risk premium of the interest rate term structures by constructing synthetic derivatives (straddles) of the interest rate swaptions in the risk-neutral measure. Their conclusions will be important when considering our results for hedging in the physical measure.

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Please insert Figure 3 here.

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The calibration of Black cap prices (Black, 1976) is produced using the Black volatility and caps/floors volatility cube method defined in Hagan and Konikov (2004). Black cap prices are then used to calibrate the volatilities of the interest rate models. The Finance Toolbox from MATLAB is used for the zero-coupon interest rate term structure estimation and the Black cap prices calibration.

5.3. Interest rate models calibration

The interest rate models are priced using binomial and trinomial tree numerical methods. The Black caps/floors volatility and Black cap prices are used for estimating the parameters of the interest rate models. In the case of the Black et al. (1990) model, $\Theta(t), \sigma(t)$ are the parameters to be estimated. $\Theta(t)$ is calibrated with the zero-coupon interest rate term structure and $\sigma(t)$ is calibrated with the Black cap prices using a least squares method, implemented in the function `capbybdt(·)` of MATLAB. In the case of the Hull and White (1990) model, the parameter $\vartheta(t)$ is calibrated with the zero-coupon interest rate term structure and $\alpha(t), \sigma(t)$ parameters are estimated from the Black prices by a least squares method implemented by the function `hwcalbycap(·)` of MATLAB.The Heath et al. (1992) model has parameters $\sigma(t,T), \alpha(t,T)$ that are calibrated with the zero-coupon interest rate term structure by the least squares method using the function `capbyhjm(·)` of MATLAB.

6. Results

Let us now analyse the empirical calibrated *CVA* of swaps, using the models of Black–Derman–Toy (*BDT*; Black et al., 1990), Black–Karasinski (*BK*; Black and Karasinski, 1991), Hull–White (*HW*; Hull and White, 1993), Heath–Jarrow–Morton (*HJM*; Heath et al., 1992), *G2++* (Brigo and Mercurio, 2003), *LIBOR* (Brace et al., 1997), and *SABR* (Hagan et al., 2002).

To obtain the term structure of interest rates we use daily data from *LIBOR* rates of up to 6 months to short term and the swap rates in the US market, with maturities between 1 and 50 years for the medium and long term. The interest rates for given maturities are interpolated. References on theoretical and empirical aspects concerning the implementation and calibration of interest rate models can be found in Bjerksund and Stensland (1996), Boyle et al. (2001), Hagan and Konikov (2004), Leippold and Wiener (2004), Galluccio et al. (2007), La Chioma and Piccoli (2007), and Keller-Ressel et al. (2012).

6.1. Empirical risk-neutral calibrated volatility

Figure 5 shows the calibrated volatility obtained by using each of the *BDT*, *HW*, *BK*, and *HJM* interest rate models. One can observe a large difference in volatility estimates obtained from distinct models of interest rates. Thus, the calibration of parameters from market data is very sensitive to the model used and can have a considerable impact on the calculation of the *CVA*.[15] *HW* and *HJM* calibrated volatilities seem to peak during the crisis near September 2008, and steadily decrease after then to the levels of 2005. *BDT* and *BK* calibrated volatilities seem to have increased since mid-2007 until recently. *BDT* and *BK* volatility behaviour is the result of the log prices modelling of interest rates, where recent lower short-term interest rates will have a higher impact than higher interest rates.

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Please insert Figure 5 here.
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Figure 6 shows the results of the calibrated volatility obtained with the *G2++*, *LIBOR*, and *SABR* interest rate models. The calibrated volatility of *G2++* is composed of two terms: $\sigma(t)$ the vol of the first factor that reproduces the volatility of the events of the 2007/2008 financial crisis and seem to stabilise after mid-2009, and $\eta(t)$ that behave as a counterpart of $\sigma(t)$ for the second linear factor, showing that the volatility of the second factor decreases during the crisis periods as evidence of a 'contagion effect' and high tail dependence, resulting in 'only one-factor' behaviour. *LIBOR* and *SABR* volatility surfaces are richer in structure and both seem to have increased since 2007/2008. The calibrated volatility surface of the *LIBOR* model is decreasing for higher maturities (higher for lower maturities); *SABR* seems to have a hump-shaped surface volatility relatively stable during the entire period, but highly volatile in the short end for the 2012/2013 period due to the decisions of the Federal Reserve over the short rate.

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In general, different interest rate models adopt certain characteristics of the volatility cube (Figure 4) extracted as in Hagan and Konikov (2004). *BDT*, *BK*, and *LIBOR* are more sensitive to the short-term rate and adjust more to the low strike (1%) caps/floors volatility, while *HW*, *HJM*, and *SABR* are balanced sensitive in all maturities and adjust more to the medium strike (5%) caps/floors volatility (Figure 4).

6.2. WWR CVA *results*

Figure 7, 8, and 9 present the 10-year interest rate swap counterparty credit risk *CVA* in the *WWR* scenario for the seven different interest rate models using the maximum positive dependence Fréchet upper bound copula. Figure 7a plots the short-term rate models' *WWR CVA* of the interest rate swap long positions

[15] In the models used, the volatility estimates are derived from at-the-money (ATM) options. However, it is important to note that different strike prices for the interest rate options involve different estimates of implied volatility. Thus, an important element in the calculation of the *CVA* also involves choosing the correct volatility model.

that yield around an average of 12% during the May 2005 to August 2013 period and start to increase by the end of 2007, reaching a maximum of 25% during the 2008 crisis and decreasing to an average level of 5% after its peak. *BDT* is the model that produces the higher *WWR CVA* estimation and *BK* is the one that produces the lowest *WWR CVA* estimation of the three models, all three models having similar behaviour before and after the crisis. In light grey we highlight the 'high volatile–larger mean' regime found by applying the Markov two-regime switching model as in (7) of Section 2.2. We can observe that the two-regime switching accurately explains the 2007/2008 credit crisis and the 2011 Euro zone Greek bailout crisis. Figure 7a plots the short-term rate models *WWR CVA* of interest rate swap short positions. In comparison to the *WWR CVA* of interest rate swap long positions, all three short-term rate models have similar behaviour before, during, and after the crisis starting from 25% before the 2007/2008 crisis and reaching the maximum at 50% shortly afterwards. An explanation is that before the crisis there was a relatively flat interest rate term structure that later got transformed into a stepped term structure after 2008 with the consequence that the interest rate swap short positions paid an extreme low short-term rate and received a higher medium- and long-term interest rate for a long time. The market perceives this anomaly in the behaviour of the interest rates and penalises it with higher counterparty credit risk, with important implications for the Federal Reserve decisions on increasing the systemic risk by holding a low level of the short-term rate.

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Figure 8 plots the 10-year interest rate swap *WWR CVA* results for the *HJM*, *LIBOR* (*LMM*), and *SABR* interest rate models. The *WWR CVA* of interest rate swap long positions (Figure 8a) averages 30–35% for the three multifactor models, from a low level of 10–20% before the 2007/2008 crisis and reaching a maximum of 45–55% shortly afterwards for *HJM* and *LMM* models, and 55% during the crisis for the *SABR* model. Similar to the short-term rate models, the behaviour of the *WWR CVA* interest rate swap long positions differs between models after the crisis,[16] since it is the *LMM* model that produces the higher *WWR CVA* estimates. The *WWR CVA* interest rate swap short position (Figure 8b) is similar for the three multifactor models, starting from 12–20% before the financial crisis to reach a maximum of 50% shortly afterwards, with *LMM* having the largest *WWR CVA* estimate. The steady high levels of *WWR CVA* after the crisis demonstrate a market premium for the Federal Reserve policy of lower levels for the short-term interest rate, increasing the interest rate swap short positions' counterparty credit risk.

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[16] Feldhütter and Lando (2008) found that the convenience yield is responsible for a great part of the interest rate swap spreads. It will be interesting to re-apply the study following the 2007/2008 financial crisis to measure the impact of credit risk and specific factors in comparison with the convenience yield weight in determining the interest rate swap spread.

Please insert Figure 8 here.
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Figure 9 shows the 10-year interest rate swap *WWR CVA* results for the *G2++* interest rate model. The *WWR CVA* of interest rate swap long positions (Figure 9a) implied by this model seems to have a two-regime behaviour: (i) one regime with an average *WWR CVA* of 10% steady during the during the sample data period from 10 May 2005 to 1 August 2013, and (ii) one regime with an average *WWR CVA* of 60–100% during mid -2005 and during the financial crisis. This is the only model that detects some counterparty risk distress during the second semester of 2005, and this could be due to the flattening of the interest rate term structure and the stopping of short-term rates increase by the Federal Reserve, signalling an initial credit risk distress. The *WWR CVA* of interest rate swap short positions (Figure 9b) reveals a similar behaviour to the *WWR CVA* of long positions with a two-regime scenario and similar values for each regime. The regime appears as a result of the exploding *G2++* first factor volatility $\sigma(t)$, as shown in Figure 6a, as the result of a contagion, and in consequence we observe that only one factor having influence on the model.

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6.3. CVA *tail dependence*

The main purpose of our empirical study is to empirically calculate the *WWR CVA* counterparty credit risk (Cherubini, 2013) of interest rate swaps and determine differences from earlier independent *PD* and *EAD* modelling (Sorensen and Bollier, 1994), and Gaussian copula dependence modelling for *CVA* calculations (Li, 2000). The dependency structure of *PD* and *EAD* is considered a tail dependency as the event of default occurs only in the lower extreme of the life distribution of the institution. Independent *CVA*, dependent *WWR CVA*, and Gaussian copula dependent *CVA* are calculated with (4), (20), and (23) respectively.

A calculation of the *CVA* Gaussian copula *PD* and *EAD* dependence model requires the estimation of an additional parameter, the time varying correlation. If *WWR* by definition is the co-monotone variation of the dependent risk factors, then we assume as the worst-case scenario a Gaussian copula correlation of $\rho = 0.9$.[17] In Figure 10 we find the estimated Gaussian copula correlation for the different windows' size ($N = 7, 8, 9, 10$) used to estimate the empirical marginal densities.[18] The $\rho(t)$ parameter of the Gaussian copula in (22) is estimated with the `copulafit(·)` function of MATLAB that calculates the sample correlation of the inverse *cdf* of the *PD* and *EAD* empirical marginal densities. The *EAD* marginal density is proxied by the average *LGD* in Table 3 of Jacobs (2012) from *Moody's Ultimate LGD Database 1987–2007*; the *PD* marginal density

[17] In a robustness check, we tested with a correlation of $\rho = 0.99$ for the Gaussian copula dependence and the interest rate swaps *WWR CVA* was still underestimated in all interest rate models.

[18] A window size of N will use the previous N year estimates' frequency as an approximation of the marginal density.

is proxied by the proportion of delisted companies in the *CRSP* database. Figure 10 shows an increasing correlation; nevertheless the estimated value never reaches the assumed worst-case scenario of $\rho(t) = 0.9$.[19]

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Interest rate swap *CVA* valuations can differ not only because of the dependency structure and the interest rate models used for its calculation, but because of the tenor of the instrument, the direction of the position (long/short), and the conditions of the market (crisis/non-crisis). Table 2 presents a panel with the results of the estimated *CVA* from 10 May 2005 to 1 August 2013 by (i) dependence structure: independent *PD* and *EAD* of long interest rate swaps $\left(CVA^{B_L}\right)$, Gaussian copula dependent *PD* and *EAD* $\left(CVA_G^{B_L}\right)$, and worst-case scenario *WWR* $\left(CVA_{WWR}^{B_L}\right)$; (ii) interest rate swap positions, long $\left(CVA^{B_L}\right)$ and short $\left(CVA^{B_S}\right)$; (iii) interest rate model: *BDT*, *HW*, *BK*, *HJM*, *G2++*,*LMM*, and *SABR*; (iv) interest rate swap tenor, short term (2 years), medium term (5 years), and long term (10 years). In Tables 3 and 4 we have the same *CVA* calculations for subsets of (i) non-crisis periods and (ii) crisis periods, respectively, applying the Markov two-regime switching model as in (7) for dividing the sets. We observe that in Tables 2, 3, and 4 there is a significant difference when using Gaussian copula and *WWR* copula dependence vs. the independent modelling of *PD* and *EAD* for all the different categories measured.

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Tables 5, 6, and 7 present the relative difference from Tables 2, 3, and 4 of the three tail dependency models for the *CVA* of interest rate swaps calculations. For the period of 10 May 2005 to 1 August 2013 the Gaussian copula dependence *CVA* is between 527 and 1609% higher than the independent *CVA* for any interest rate model, interest rate swap tenor, and instrument position, and the *WWR* is about 51–362% larger than the Gaussian copula. For the crisis period, although the *CVA* estimates are higher, the relative

[19] Liu et al. (2015) provides a method for calculating confidence intervals of tail dependence measures that could be used to provide an interval for our implicit time varying correlation.

difference is lower, and during the non-crisis period the relative difference is higher; this inverse effect could be due to the fact that the distributions of the marginals of the *PD* and *EAD* during the crisis have extreme values that co-variate more closely to the Gaussian copula dependence structure; however, the difference remains significant.

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The most important factor when measuring the *CVA* of interest rate swaps was the dependency structure, followed by the tenor and the interest rate model. The results are significant when measuring counterparty credit risk for the actual regulatory capital framework, financial institutions, and risk managers.

6.4. Robustness checks

Due to the complexity implied in the valuation of the *CVA* of interest rate swaps, we conducted several robustness checks; given our space constraints, we summarise them below without full detail:

(i) We find that the estimated *LGD* varies with time and has a strong correlation with the *EAD*. We produced a new study on which the default intensity $\lambda(t_i)$ at time t_i is modelled by:

$$\lambda(t_i) = \frac{s(t_i)}{LGD} = 0.4, \tag{39}$$

where $s(t_i)$ denotes the credit spread at time t_i.[20] In this new study the ratio of the spread and the *LGD* will be constant, introducing a independent factor of a constant default intensity into the *CVA* calculations to analyse its effect in the tail dependency behaviour. The results demonstrate that the relative difference when modelling tail dependence between *PD* and *EAD* for the *CVA* of interest rate swaps is reduced for the three different dependency models analysed: the estimated *CVA* with Gaussian copula dependence is 170–307% higher than the *CVA* with independent *PD* and *EAD*,

[20] We calculated the mean default rates of Moody's Caa-C rated companies (Moody's Investor Service, 2011) yielding 27.68% with a maximum of 57.89 for 1990; we then defined a default intensity of 0.4 considered an extreme event, with 40% of the companies defaulting in one year.

compared to the original 527–1609% estimates with time varying default intensity, for the period from 10 May 2005 to 1 August 2013. The relative difference between the *WWR CVA* and the *CVA* with Gaussian copula reduces to 0.74–160% from 51–362% in the original estimation with time varying default intensity. Notwithstanding, there exist significant differences in all categories between each of the three dependency models, emphasising the importance of tail dependency modelling.

(ii) The Gaussian copula dependence model required to provide a correlation parameter $\rho(t)$. We initially defined $\rho(t) = 0.9$ as this is the largest empirical correlation value found when we estimated the copula correlation (Figure 10, Section 6.3). In this robustness check, we defined the Gaussian copula constant correlation of $\rho(t) = 0.99$. The results demonstrate a significant difference between the *WWR CVA* and the *CVA* with Gaussian copula modelling of more than 100% in many cases.

(iii) We analysed the calculated the independent *CVA*, Gaussian copula dependent *CVA*, and the *WWR CVA* surfaces (interest rate swap tenor vs. date) for the seven interest rate models studied, for long and short interest rate swap positions. We further analysed the differences between the *WWR CVA* long minus the *WWR CVA* of short interest rate swap positions. We found no observable difference between the seven interest rate models, with two exceptions: the two-regime behaviour of the *G2++* model, and the difference between the *WWR CVA* of long minus short interest rate swap positions of the *LIBOR* model. In the case of the G2++ model, this behaviour was previously observed and is the result of one-factor contagion dominating the behaviour of the model. In the case of the inverted *WWR CVA* long minus short *LIBOR* model behaviour, for all models except *LIBOR*, extreme low short-term interest rates favour a high premium for the counterparty credit risk of short positions in interest rate swaps holders, as they will receive a positive difference fuelled by the low-rate policy of the Federal Reserve, generating a negative surface for this difference (the *WWR CVA* of short positions is higher for non-market models, as they neglect the market expectations of the forward curve); however, the *LIBOR* model accounts for the forward expectations through market prices, and holders of long positions of interest rate swaps have been faced in this model with a steep yield curve since the 2007/2008 crisis. This steep yield curve is priced in the premium of counterparty credit risk long positions (the (*WWR CVA* of long positions is higher for market models).

(iv) We tested additional sensitivity analysis for the different parameters of the *CVA* calculations of interest rate swaps, valued with the *HW* model. The effects of the local volatility $\sigma(t)$ of the swaption model, the linear combination ρ between the independent and the *WWR CVA*, the interest rate swap tenor, and shifts in the interest rate yield curve were analysed for different specific values. We found that the volatility increases the magnitude of the *CVA*; a linear combination that favours the independent model will steep the medium-term *CVA* curve and reduce the long-term *CVA* curve; an increase in the tenor will produce a humped *CVA* curve with higher values in the mid-term portion of the curve; and an interest rate shift will negligible increase the *CVA* value.

7. Conclusions

In this research we tested the importance of dependency in credit risk factors, such as the probability of default *PD* and exposure at default *EAD*. Our approach involves the use of copulae linking two components of credit risk. Considering a dependency between the instant of default and the exposure value and using a copula that establishes a perfect dependence, Cherubini (2013) develops an analytical formula for the calculation of the *CVA* of interest rate swaps.

The use of copulas allows the study of *WWR* enabling the incorporation of dependency that was often overlooked, as shown in the simplified formulas described, for example, in Canabarro (2009) and Gregory (2012). In this context, this paper sought to extend the analysis of Cherubini (2013), taking into account different models of interest rate diffusion. Considering that the pricing of swaps depends on the modelling of the term structure of interest rates and their behaviour over time until the maturity of a derivative, the analysis is justified, as different assumptions can have significant impact on the estimates of *CVA*.

Applying interest rate models from Black et al. (1990), Hull and White (1990), Black and Karasinski (1991), Heath et al. (1992), the *G2++* of Brigo and Mercurio (2003), the *LIBOR* of Brace et al. (1997), and the *SABR* of Hagan et al. (2002), we calculate the *CVA* of interest rate swaps using the interest rate swaption implied market volatility for the period of 10 May 2005 to 1 August 2013, and comparatively analyse the estimates of the three different models of credit risk factors dependency for the *CVA* calculations (independent *PD* and EAD, Gaussian copula dependence, and *WWR* copula dependence). The differences between the three different dependency models range from 51% to 1609% of the estimated *CVA*, revealing a clear underestimation when using regulatory independent models. The results indicate the existence of differences that can impact the price adjustments in portfolios of derivatives, although no interest rate model consistently generates higher or lower values of the *CVA*, suggesting a difficulty in reducing the capital requirement solely from the choice of the diffusion model of interest rates.

These results have an important consequence for the regulatory institutions and risk managers. From the point of view of capital adequacy, whether there are models that underestimate the *CVA* can be beneficial for the purpose of capital requirements to meet regulatory agency demands, although it increases the likelihood of non-conformities, which can later result in an increase in capital requirements. From the point of view of risk management, in the case of swaps, the dependence of the *CVA* on the choice of interest rate, swaptions and volatility models implies the existence of risk modelling. Thus, the analysis of various dependency models is essential for a proper assessment of the *CVA*, and we suggest as for future research multivariate dependency models that include all credit risk factors.

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A. Mathematical Proofs

A.1. Proof of Proposition 2.1

Proof. The discussion follows closely Brigo et al. (2013b). To prove the equivalence between (2) and (3), we consider the definition of CVA given by Brigo et al. (2013b):

$$\begin{aligned} CVA_C^B &= \mathbb{E}_t[\Pi(t,T) - \tilde{\Pi}_(t,T)] \\ &= \mathbb{E}_t\Big[\Pi(t,T) - \mathbb{1}_{(\tau_c>T)}\Pi(t,T) - \\ &\mathbb{1}_{(t<\tau_C\leq T)}\Big[\Pi(t,\tau_C) + D(t,\tau_C)\Big(\min(NPV(\tau_C),0) + RR_C\max(NPV(\tau_C),0)\Big)\Big]\Big]. \end{aligned} \tag{40}$$

Given that $\Pi(t,T) = \mathbb{1}_{(t<\tau_C\leq T)}\Pi(t,T) + \mathbb{1}_{(\tau_C>T)}\Pi(t,T)$, then:

$$\begin{aligned} CVA_C^B = \mathbb{E}_t\Big[&\mathbb{1}_{(t<\tau_C\leq T)}\Pi(t,T) - \mathbb{1}_{(t<\tau_c\leq T)}\Big[\Pi(t,\tau_C) + D(t,\tau_C) \\ &\Big(\min(NPV(\tau_C),0) + RR_C\max(NPV(\tau_C),0)\Big)\Big]\Big]. \end{aligned} \tag{41}$$

Defining NPV^+ and NPV^- as the positive and negative parts of NPV, respectively, we have $NPV^+(\tau_C) = \max(NPV(\tau_C),0)$ e $NPV^-(\tau_C) = -\min(NPV(\tau_C),0)$, which leads to:

$$\begin{aligned} CVA_C^B = \mathbb{E}_t\Big[&\mathbb{1}_{(t<\tau_C\leq T)}\Pi(t,T) - \mathbb{1}_{(t<\tau_C\leq T)}\Big[\Pi(t,\tau_C) + D(t,\tau_C) \\ &\Big(-NPV^-(\tau_C) + RR_C NPV^+(\tau_C)\Big)\Big]\Big]. \end{aligned} \tag{42}$$

Since $NPV(\tau_C) = NPV^+(\tau_C) - NPV^-(\tau_C)$, then $-NPV^-(\tau_C) = NPV(\tau_C) - NPV^+(\tau_C)$. Following Brigo et al. (2013b), therefore:

$$\begin{aligned} CVA_C^B &= \mathbb{E}_t\Big[\mathbb{1}_{(t<\tau_C\leq T)}\Pi(t,T) - \mathbb{1}_{(t<\tau_C\leq T)}\Big[\Pi(t,\tau_C) + D(t,\tau_C) \\ &\qquad \Big(NPV(\tau_C) - NPV^+(\tau_C) + RR_C NPV^+(\tau_C)\Big)\Big]\Big] \\ &= \mathbb{E}_t\Big[\mathbb{1}_{(t<\tau_C\leq T)}\Pi(t,T) - \mathbb{1}_{(t<\tau_C\leq T)}\Big[\Pi(t,\tau_C) + D(t,\tau_C)NPV(\tau_C) + D(t,\tau_C) \\ &\qquad \Big(-NPV^+(\tau_C) + RR_C NPV^+(\tau_C)\Big)\Big]\Big] \\ &= \mathbb{E}_t\Big[\mathbb{1}_{(t<\tau_C\leq T)}\Pi(t,T) - \mathbb{1}_{(t<\tau_C\leq T)}\Big[\Pi(t,\tau_C) + D(t,\tau_C)NPV(\tau_C)\Big] + \mathbb{1}_{(t<\tau_C\leq T)}\Big[D(t,\tau_C) \\ &\qquad \Big(NPV^+(\tau_C)(1+RR_C)\Big)\Big]\Big]. \end{aligned} \tag{43}$$

Since $\Pi(t,T) = \Pi(t,\tau_C) + D(t,\tau_C)NPV(\tau_C)$ and taking into account the definitions of $LGD_C = 1 - RR_C$ and $EAD_C = (NPV(\tau_C))^+$ previously stated, then:

$$CVA_C^B = \mathbb{E}_t\Big[\mathbb{1}_{(t<\tau_C\leq T)} D(t,\tau_C) NPV^+(\tau_C)(1+RR_C)\Big] = \mathbb{E}_t\Big[\mathbb{1}_{(t<\tau_C\leq T)} \cdot LGD_C \cdot EAD_C)\Big]. \tag{44}$$

□

A.2. Proof of Proposition 3.1

Proof. The discussion follows closely Cherubini (2013). Then, the *CVA* of counterparty B subject to the counterparty credit risk of C can be defined as:

$$CVA_C^B = LGD_C \cdot \sum_{j=1}^{n} A\left(t, t_j, t_n\right) \int_{r_s(t_0,t_n)}^{\infty} \min\left[1 - \mathbb{Q}\left(u\right), \mathbb{S}_B\left(t_{j-1}\right) - \mathbb{S}_B\left(t_j\right)\right] \mathrm{d}\, u. \tag{45}$$

Notice that $\mathbb{S}_C\left(t_{j-1}\right) - \mathbb{S}_B\left(t_j\right) \leq 1$ does not depend on u. In addition, $1 - \mathbb{Q}\left(u\right)$ is monotonically decreasing. Thus, two cases can be analysed: (1) $1 - \mathbb{Q}\left(u\right) \leq \mathbb{S}_C\left(t_{j-1}\right) - \mathbb{S}_B\left(t_j\right)$ and (2) $1 - \mathbb{Q}\left(u\right) > \mathbb{S}_C\left(t_{j-1}\right) - \mathbb{S}_B\left(t_j\right)$.

In the first case, we have:

$$\lim_{u\to\infty}\left(1 - \mathbb{Q}(u)\right) \to 0 \leq 1 - \mathbb{Q}\left(r_s(t_0,t_n)\right) \leq \mathbb{S}_C\left(t_{j-1}\right) - \mathbb{S}_C\left(t_j\right). \tag{46}$$

Thus, the integral in (45) is equivalent to:

$$CVA_C^B = LGD_C \cdot \sum_{j=1}^{n} A(t,t_j,t_n) \cdot \int_{sr(t_0,t_n)}^{\infty} \left(1 - \mathbb{Q}\left(u\right)\right) \mathrm{d}\, u. \tag{47}$$

In the second case, we have:

$$\lim_{u\to\infty}\left(1 - \mathbb{Q}(u)\right) \to 0 \leq \mathbb{S}_C\left(t_{j-1}\right) - \mathbb{S}_C\left(t_j\right) < 1 - \mathbb{Q}\left(r_s(t_0,t_n)\right). \tag{48}$$

In this case, the integral in (45) can be calculated in two parts, denoting scenarios between $[r_s(t_0,t_n), z)$ and $[z,\infty)$. Thus,

$$CVA_C^B = LGD_C \cdot \sum_{j=1}^{n} A(t,t_j,t_n) \cdot \left(\int_{r_S(t_0,t_n)}^{z} \left(\mathbb{S}_C\left(t_{j-1}\right) - \mathbb{S}_C\left(t_j\right) \mathrm{d}\, u\right) + \int_{z}^{\infty} \left(1 - \mathbb{Q}\right)(u)\, \mathrm{d}\, u \right), \tag{49}$$

with $z \in [r_S(t_0,t_n), \infty)$ e $\{z : \mathbb{S}_C\left(t_{j-1}\right) - \mathbb{S}_C\left(t_j\right) = 1 - \mathbb{Q}\left(z\right)\}$.

Defining $k(t_j) = f^{-1}\left(\mathbb{S}_C\left(t_{j-1}\right) - \mathbb{S}_C\left(t_j\right)\right)$, with $f = 1 - \mathbb{Q}(u)$, then:

$$CVA_C^B = LGD_C \cdot \sum_{j=1}^{n} A(t, t_j, t_n) \cdot \left(\int_{k(t_j)}^{\infty} (1 - \mathbb{Q}(u)) \, \mathrm{d}\, u + k(t_j) - r_s(t_0, t_n))(\mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j)) \right). \quad (50)$$

Let use (47) and (50), for the *CVA* calculation in (45) to yield the result.

Conversely, one can analyse the *CVA* of a short position in a swap, in which the B bank receives values indexed to a fixed interest rate and pays a floating interest rate. In this case, the *WWR* involves the probability of default increasing while the market interest rate falls. Thus, the bank pays less in the short position, getting a higher fixed interest rate, obtaining gain in its marked-to-market position, however incurring in a counterparty credit risk. Thus, in the case of a short position in the swap, the agent is subject to *WWR* when there is a negative association between swap rates and the probability of default of the counterparty.

Therefore, taking into consideration Cherubini (2013), one extreme situation involves:

$$\mathbb{C}(\mathbb{Q}(u), \mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j)) = \min(\mathbb{Q}(u), \mathbb{S}_C(t_{j-1}) - \mathbb{S}_C(t_j)). \quad (51)$$

Using an analogous argument to the long position in the swap, for the short position, one has:

$$CVA_C^B = LGD_C \cdot \sum_{j=1}^{n} A(t, t_j, t_n) \cdot \int_0^{r_s(t_0, t_n)} \min[\mathbb{Q}(u), \mathbb{S}_C(t_{j-1}) - \mathbb{S}_B(t_j)] \, \mathrm{d}\, u. \quad (52)$$

Defining $k^*(t_j) = g^{-1}\left(\mathbb{S}_C\left(t_{j-1}\right) - \mathbb{S}_C\left(t_j\right)\right)$, with $g = \mathbb{Q}(u)$, and we follow Cherubini (2013) to yield the result.

□

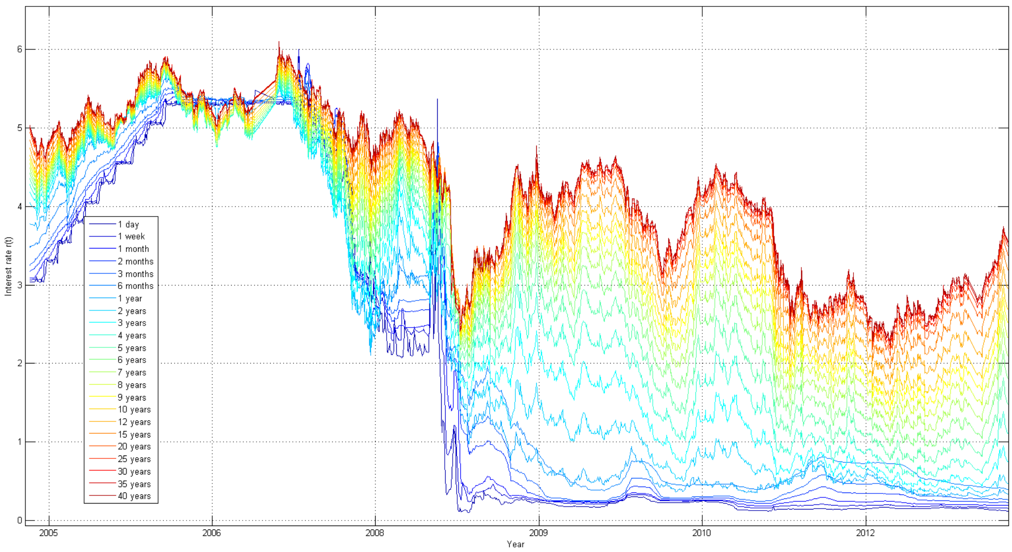


Figure 1: This figure shows the *LIBOR* and swap rates evolution from May 2005 to August 2013. The historical data of *LIBOR* rates go from spot to 6-month maturities. The swap rates are from 1 year to 40-year maturities. The rates are extracted from Bloomberg.

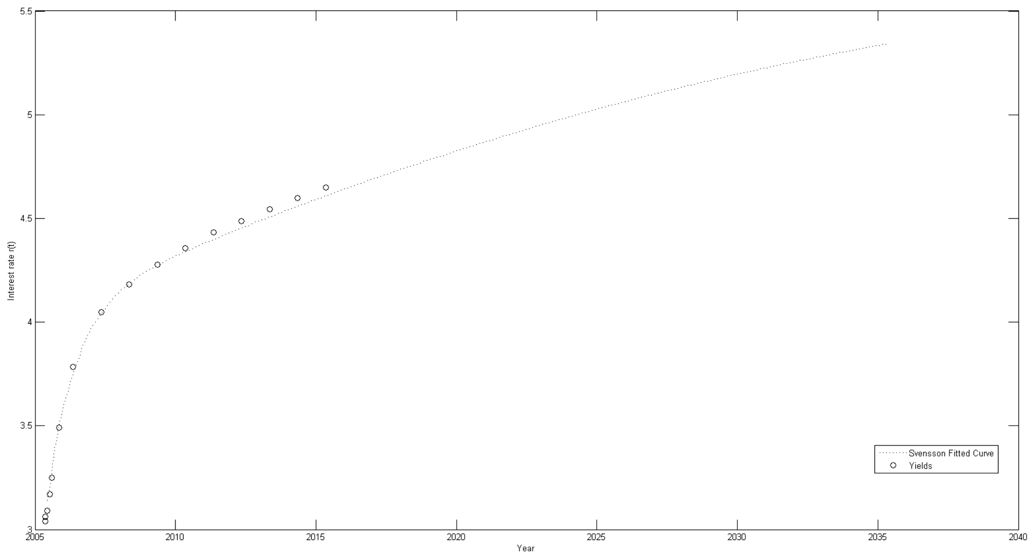


Figure 2: This figure shows the interest rate forward term structure calibrated using the Nelson–Siegel–Svensson model for 10 May 2005. The actual forward rates are in circles. The rates are extracted from Bloomberg.

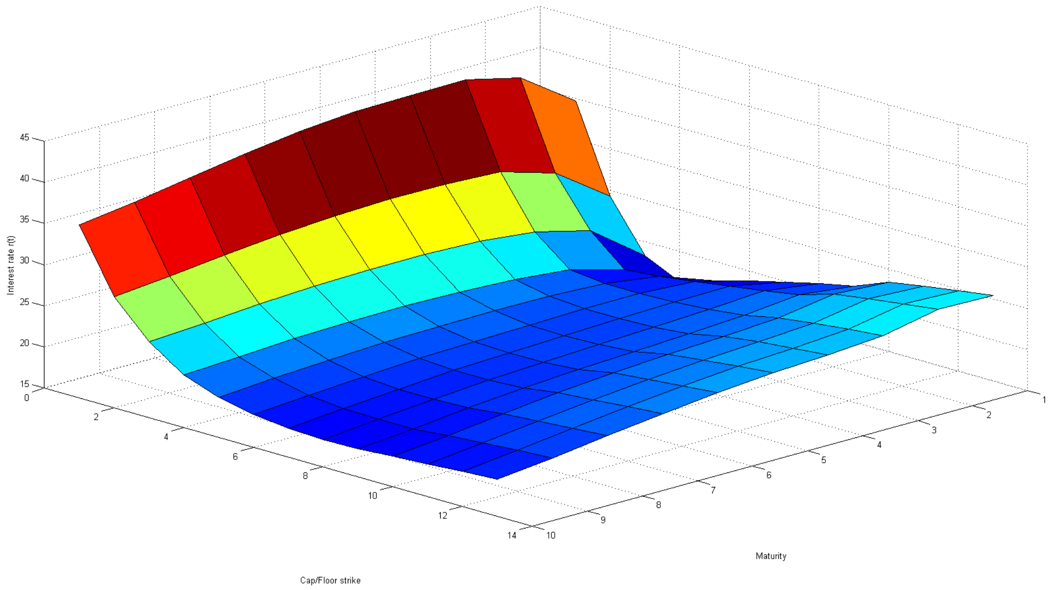
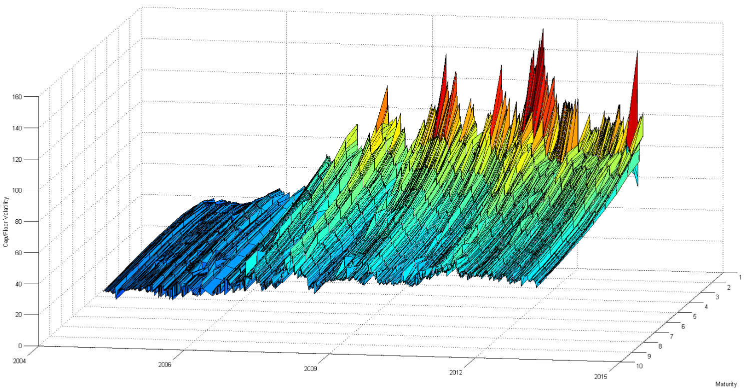
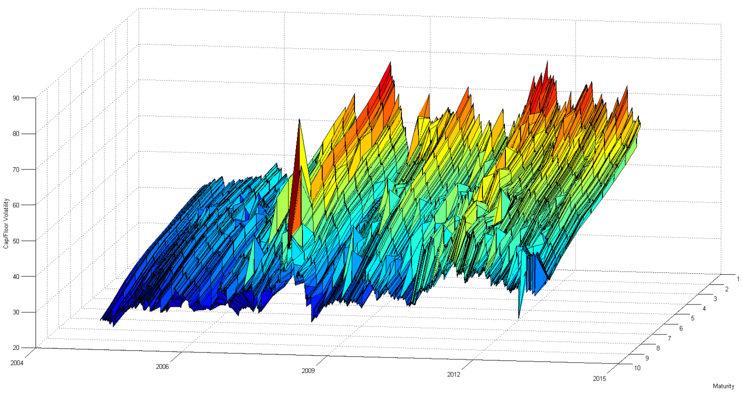


Figure 3: Volatility surface of caps/floors for the period of 10 May 2005.

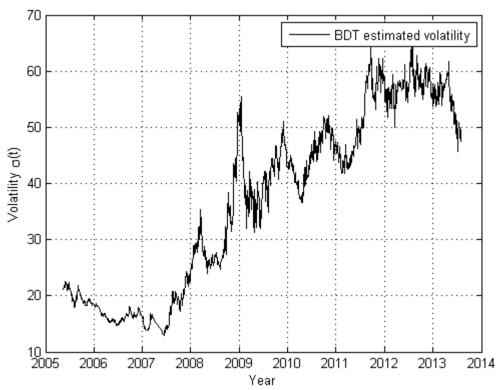


(a) Volatility surface of 1% strike caps/floors from 10 May 2005 to 1 August 2013.

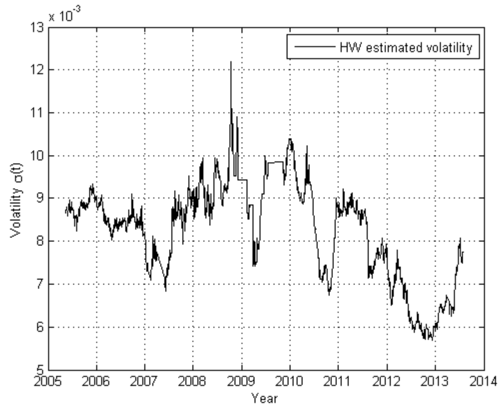


(b) Volatility surface of 5% strike caps/floors from 10 May 2005 to 1 August 2013.

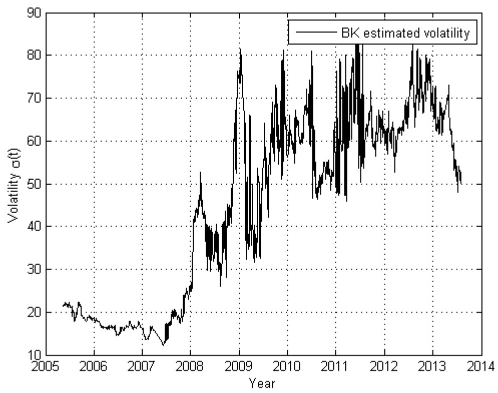
Figure 4: This figure shows the caps/floors volatility surface for the period of 10 May 2005 to 1 August 2013.



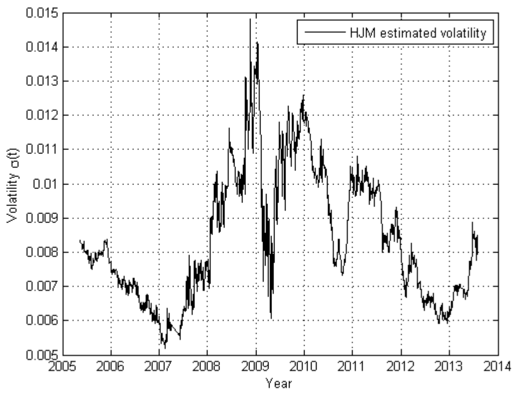
(a) *BDT* calibrated volatility.



(b) *HW* calibrated volatility.

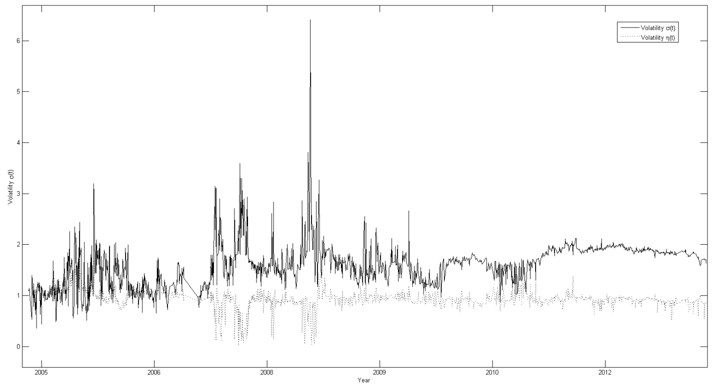


(c) *BK* calibrated volatility.

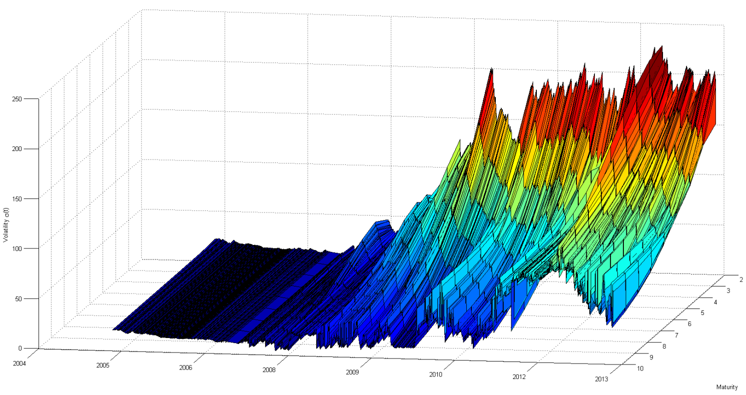


(d) *HJM* calibrated volatility.

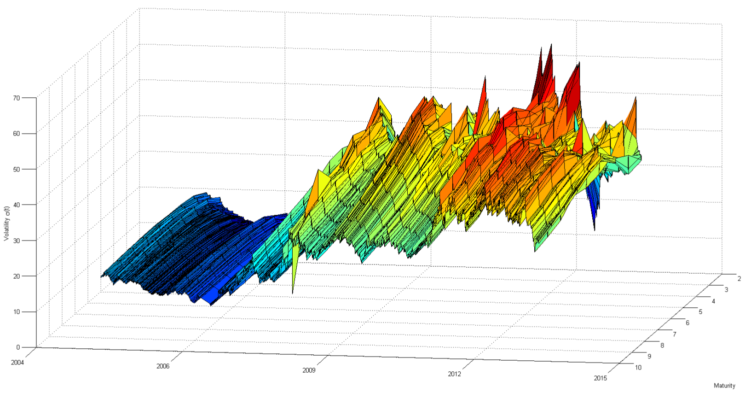
Figure 5: This figure shows the calibrated volatility for four different interest rate models (*BDT, HW, BK, HJM*).



(a) *G2++* calibrated volatility.

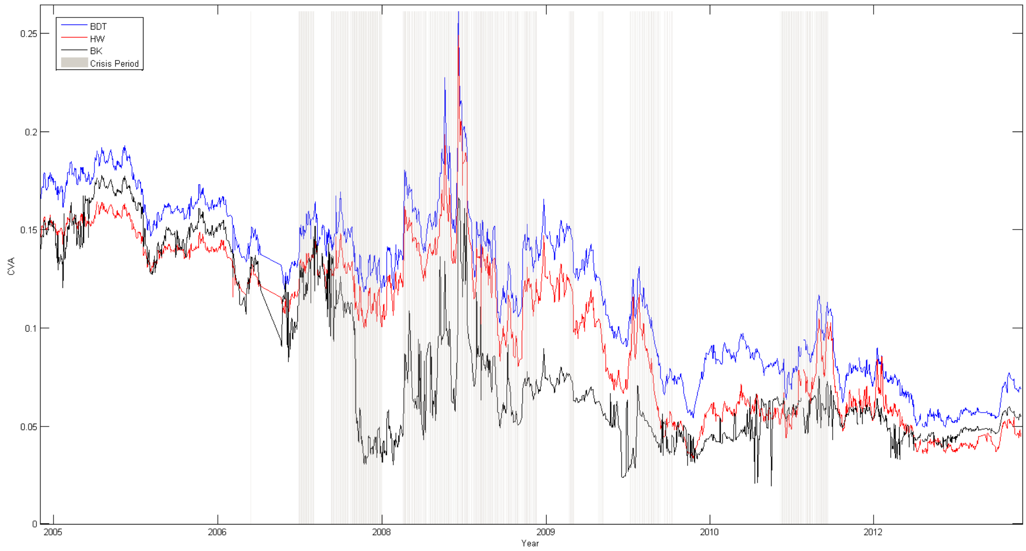


(b) *LMM* calibrated volatility.

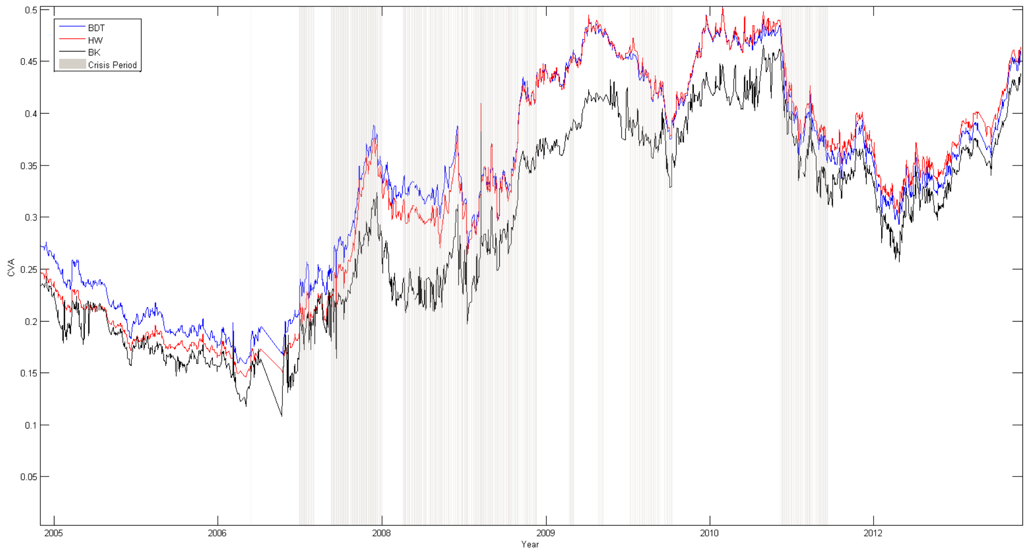


(c) *SABR* calibrated volatility.

Figure 6: This figure shows the calibrated volatility function and surface for the two-factor *G2++*, *LIBOR*, and *SABR* interest rate models.



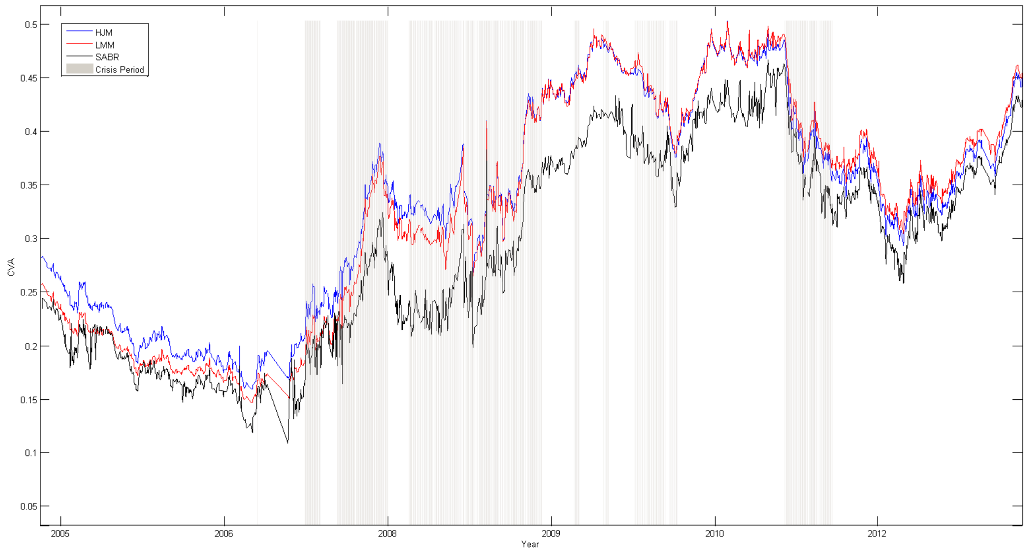
(a) The Credit Valuation Adjustment (CVA) for a long position in an Interest Rate Swap (IRS).



(b) The Credit Valuation Adjustment (CVA) for a short position in an Interest Rate Swap (IRS).

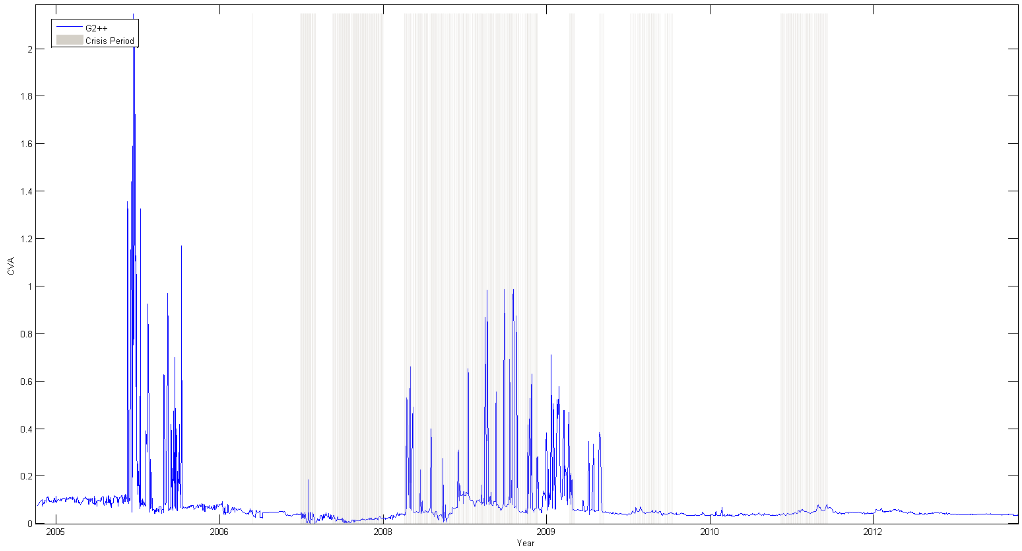
Figure 7: This figure shows the Credit Valuation Adjustment (CVA) in an Interest Rate Swap (IRS) with one-factor short-rate models: Black, Derman, and Toy (BDT), Hull and White (HW), and Black and Karasinski (BK) in a two-regime volatility framework (crisis period – high volatility in grey).

(a) The Credit Valuation Adjustment (CVA) for a long position in an Interest Rate Swap (IRS).

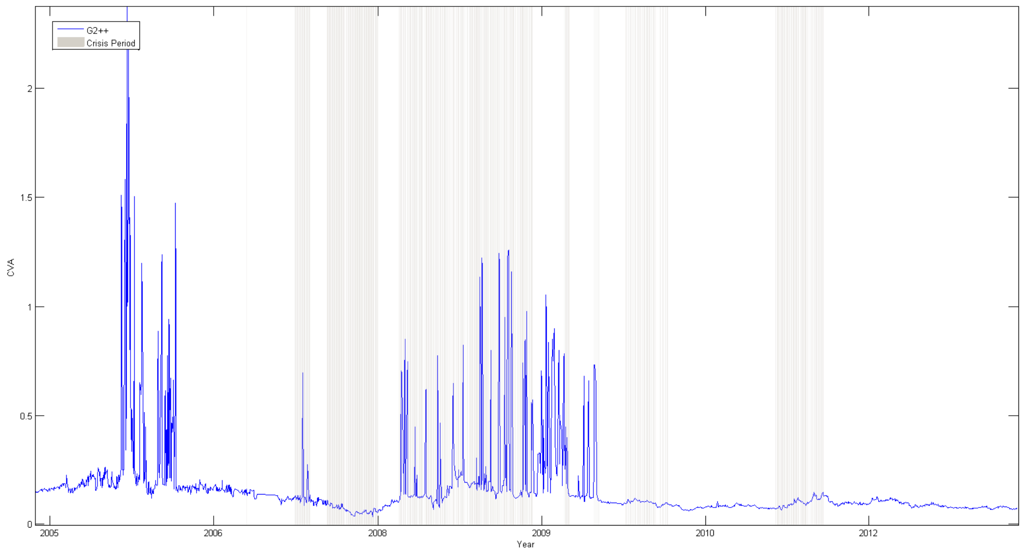


(b) The Credit Valuation Adjustment (CVA) for a short position in an Interest Rate Swap (IRS).

Figure 8: This figure shows the Credit Valuation Adjustment (CVA) in an interest rate swap with multifactor-factor interest rate models: Heath, Jarrow, and Morton (HJM), $LIBOR$ market model (LMM), and stochastic alpha, rho, and beta ($SABR$) in a two-regime volatility framework (crisis period – high volatility in grey).



(a) The Credit Valuation Adjustment (CVA) for a long position in an Interest Rate Swap (IRS).



(b) The Credit Valuation Adjustment (CVA) for a short position in an Interest Rate Swap (IRS).

Figure 9: This figure shows the Credit Valuation Adjustment (CVA) in an Interest Rate Swap (IRS) for the two-factor $G2{+}{+}$ interest rate model in a two-regime volatility framework (crisis period - high volatility in grey).

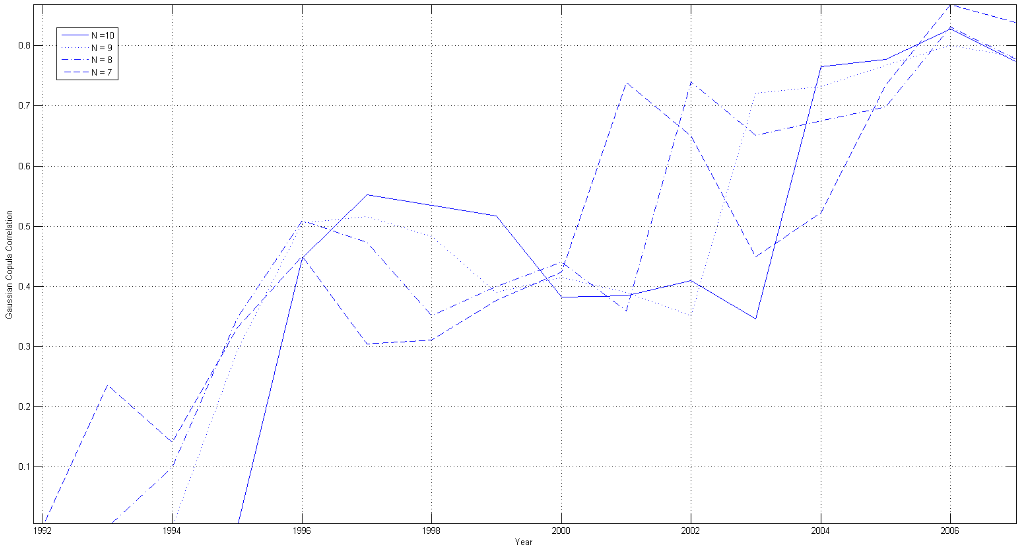


Figure 10: This figure shows the estimated Gaussian copula dependence correlation ($\rho(t)$) parameter for different window sizes ($N = 7, 8, 9, 10$) of the frequencies used to proxy the marginal empirical distribution of PD and EAD.

Table 1: Interest Rate Market Data

This table displays the different fixed-income instruments used for calculating the interest rate term structure curve used in the valuation of interest rate swaps (IRS) and swaptions.

Interest rate description	Tenor	Bloomberg Code
US dollar *LIBOR*	1 day	US00O/N Index
	1 week	US0001W Index
	1 month	US0001M Index
	2 month	US0002M Index
	3 month	US0003M Index
	6 month	US0006M Index
US dollar swap rate	1 year	USSW1 Curncy
	2 years	USSW2 Curncy
	3 years	USSW3 Curncy
	4 years	USSW4 Curncy
	5 years	USSW5 Curncy
	6 years	USSW6 Curncy
	7 years	USSW7 Curncy
	8 years	USSW8 Curncy
	9 years	USSW9 Curncy
	10 years	USSW10 Curncy
	12 years	USSW12 Curncy
	15 years	USSW15 Curncy
	20 years	USSW20 Curncy
	25 years	USSW25 Curncy
	30 years	USSW30 Curncy
	35 years	USSW35 Curncy
	40 years	USSW40 Curncy

Table 2: The Credit Valuation Adjustment of Interest Rate Swaps

This table displays the counter party credit risk (CCR) measured by the credit valuation adjustment (CVA) of long–short positions in interest rate swaps (IRS) of different maturities (2, 5, and 10 years). The CVA is calculated for the period of 10 May 2005 to 1 August 2013. Six different interest rate models were used for the calculations: Black, Derman and Toy (BDT), Hull and White (HW), Black and Karasinski (BK), Heath, Jarrow, and Morton (HJM), Two-Factor Gaussian (G2++), and SABR models. Rates are in percentages (%). The interest rates for calculations were extracted from Bloomberg.

		Counter Party Credit Risk					
		Long (*IRS* Fixed-rate Payer)			Short (*IRS* Float-rate Payer)		
		CVA^{B_L}	$CVA_G^{B_L}$	$CVA_{WWR}^{B_L}$	CVA^{B_S}	$CVA_G^{B_S}$	$CVA_{WWR}^{B_S}$
Short-term *IRS* (2-years)	**BDT**	0.07	1.05	2.12	0.55	4.63	11.32
	HW	0.09	1.34	2.19	0.57	4.68	11.39
	BK	0.04	0.69	1.47	0.52	4.48	10.11
	HJM	0.08	1.14	2.25	0.56	4.66	11.59
	G2++	0.24	2.70	4.38	0.09	1.48	6.82
	LMM	0.42	4.06	8.03	0.29	3.28	9.28
	SABR	0.09	1.27	2.36	0.54	4.55	11.20
Mid-term *IRS* (5-year)	**BDT**	0.39	3.56	7.24	1.23	9.12	24.45
	HW	0.40	3.89	6.57	1.25	9.17	24.37
	BK	0.26	2.59	4.97	1.11	8.60	21.42
	HJM	0.37	3.52	6.60	1.22	9.08	24.53
	G2++	0.46	4.50	6.84	0.37	3.90	12.86
	LMM	0.88	7.56	13.93	0.88	7.30	18.22
	SABR	0.45	4.04	9.83	1.26	9.14	24.89
Long-term *IRS* (10-year)	**BDT**	0.66	4.75	12.23	1.58	10.58	32.46
	HW	0.66	5.00	10.27	1.57	10.55	31.84
	BK	0.49	3.61	8.67	1.41	9.91	28.30
	HJM	0.61	4.59	10.17	1.53	10.43	31.86
	G2++	0.57	4.94	8.60	0.78	5.38	16.81
	LMM	1.08	8.47	16.85	1.46	9.19	22.87
	SABR	0.72	5.24	16.45	1.60	10.59	32.86

Table 3: The Credit Valuation Adjustment of Interest Rate Swaps (Crisis Period)

This table displays the counter party credit risk (CCR) measured by the credit valuation adjustment (CVA) of long–short positions in interest rate swaps (IRS) of different maturities (2, 5, and 10 years). The CVA is calculated only for the crisis period as the grey area in Figure 7, between 10 May 2005 to 1 August 2013. Six different interest rate models were used for the calculations: Black, Derman and Toy (BDT), Hull and White (HW), Black and Karasinski (BK), Heath,Jarrow, and Morton (HJM), Two-Factor Gaussian (G2++), and SABR models. Rates are in percentages (%). The interest rates for calculations were extracted from Bloomberg.

		Counter Party Credit Risk					
		Long (*IRS* Fixed-rate Payer)			**Short (*IRS* Float-rate Payer)**		
		CVA^{B_L}	$CVA_G^{B_L}$	$CVA_{WWR}^{B_L}$	CVA^{B_S}	$CVA_G^{B_S}$	$CVA_{WWR}^{B_S}$
Short-term *IRS* (2-years)	**BDT**	0.11	1.43	2.09	0.62	5.21	11.98
	HW	0.13	1.76	2.40	0.64	5.28	12.17
	BK	0.06	0.85	1.07	0.56	4.96	10.36
	HJM	0.12	1.57	2.32	0.63	5.25	12.26
	G2++	0.35	3.36	4.81	0.10	1.49	7.52
	LMM	0.50	4.65	8.27	0.30	3.40	9.75
	SABR	0.16	1.91	3.12	0.64	5.29	12.31
Mid-term *IRS* (5-year)	**BDT**	0.53	4.59	7.59	1.40	10.21	26.06
	HW	0.54	4.88	7.21	1.41	10.22	25.89
	BK	0.34	3.14	3.89	1.21	9.37	21.88
	HJM	0.53	4.66	6.92	1.40	10.21	26.18
	G2++	0.69	5.68	7.16	0.39	3.82	13.81
	LMM	1.06	8.60	14.22	0.90	7.51	19.43
	SABR	0.65	5.50	12.04	1.49	10.51	27.52
Long-term *IRS* (10-year)	**BDT**	0.87	5.98	13.23	1.80	11.80	34.73
	HW	0.85	6.16	11.43	1.77	11.72	33.83
	BK	0.60	4.27	7.51	1.52	10.72	28.84
	HJM	0.83	5.91	11.01	1.76	11.68	34.05
	G2++	0.84	6.20	8.53	0.80	5.28	17.22
	LMM	1.29	9.61	17.24	1.51	9.48	24.38
	SABR	0.98	6.87	19.42	1.88	12.09	36.07

Table 4: The Credit Valuation Adjustment of Interest Rate Swaps (Non-crisis Period)

This table displays the counter party credit risk (CCR) *measured by the credit valuation adjustment* (CVA) *of long–short positions in interest rate swaps* (IRS) *of different maturities (2, 5, and 10 years). The* CVA *is calculated only for the crisis period as the non-grey area in Figure 7, between 10 May 2005 to 1 August 2013. Six different interest rate models were used for the calculations: Black, Derman and Toy* (BDT)*, Hull and White* (HW)*, Black and Karasinski* (BK)*, Heath,Jarrow, and Morton* (HJM)*, Two-Factor Gaussian* (G2++)*, and* SABR *models. Rates are in percentages (%). The interest rates for calculations were extracted from Bloomberg.*

		Counter Party Credit Risk					
		Long (***IRS*** Fixed-rate Payer)			Short (***IRS*** Float-rate Payer)		
		CVA^{B_L}	$CVA_G^{B_L}$	$CVA_{WWR}^{B_L}$	CVA^{B_S}	$CVA_G^{B_S}$	$CVA_{WWR}^{B_S}$
Short-term *IRS* (2-years)	**BDT**	0.06	0.90	2.13	0.52	4.40	11.05
	HW	0.07	1.17	2.11	0.54	4.43	11.06
	BK	0.03	0.62	1.63	0.49	4.29	10.00
	HJM	0.06	0.97	2.22	0.52	4.42	11.31
	G2++	0.19	2.43	4.21	0.09	1.47	6.53
	LMM	0.38	3.81	7.93	0.28	3.24	9.10
	SABR	0.07	1.00	2.05	0.50	4.25	10.74
Mid-term *IRS* (5-year)	**BDT**	0.33	3.13	7.10	1.16	8.68	23.79
	HW	0.35	3.48	6.31	1.18	8.74	23.74
	BK	0.23	2.37	5.41	1.07	8.29	21.23
	HJM	0.31	3.06	6.47	1.15	8.62	23.85
	G2++	0.37	4.02	6.71	0.36	3.93	12.47
	LMM	0.81	7.13	13.81	0.87	7.21	17.72
	SABR	0.37	3.45	8.92	1.16	8.58	23.81
Long-term *IRS* (10-year)	**BDT**	0.58	4.25	11.81	1.50	10.08	31.53
	HW	0.58	4.53	9.79	1.49	10.08	31.02
	BK	0.44	3.35	9.14	1.36	9.57	28.08
	HJM	0.52	4.05	9.82	1.44	9.92	30.96
	G2++	0.45	4.42	8.63	0.77	5.43	16.64
	LMM	0.99	8.00	16.69	1.45	9.07	22.26
	SABR	0.62	4.58	15.23	1.49	9.98	31.55

Table 5: Difference in Credit Valuation Adjustment Calculations Between Dependency Models

*This table displays the difference in counter party credit risk (*CCR*) calculations for different dependency models (independent, Gaussian, maximum), when measured by the credit valuation adjustment (*CVA*) of long–short positions in interest rate swaps (*IRS*) of different maturities (2, 5, and 10 years). The* CVA *is calculated for the period of 10 May 2005 to 1 August 2013. Six different interest rate models were used for the calculations: Black, Derman and Toy (*BDT*), Hull and White (*HW*), Black and Karasinski (*BK*), Heath,Jarrow, and Morton (*HJM*), Two-Factor Gaussian (*G2++*), and* SABR *models. Rates are in percentages (%). The interest rates for calculations were extracted from Bloomberg.*

		Counter Party Credit Risk			
		Long (***IRS*** Fixed-rate Payer)		Short (***IRS*** Float-rate Payer)	
		CVA^{B_L} **vs.** $CVA_G^{B_L}$	$CVA_G^{B_L}$ **vs.** $CVA_{WWR}^{B_L}$	CVA^{B_S} **vs.** $CVA_G^{B_S}$	$CVA_G^{B_S}$ **vs.** $CVA_{WWR}^{B_S}$
Short-term *IRS* (2-years)	**BDT**	1317.01%	100.78%	743.78%	144.39%
	HW	1371.82%	63.30%	726.50%	143.44%
	BK	1609.87%	112.20%	770.52%	125.40%
	HJM	1325.79%	96.36%	739.31%	148.61%
	G2++	1041.67%	62.21%	1543.16%	362.36%
	LMM	876.29%	97.78%	1039.03%	182.73%
	SABR	1260.72%	86.04%	736.83%	145.83%
Mid-term *IRS* (5-year)	**BDT**	820.15%	103.70%	639.48%	168.00%
	HW	864.65%	68.92%	633.20%	165.70%
	BK	894.39%	91.61%	676.25%	149.03%
	HJM	843.56%	87.45%	643.58%	170.05%
	G2++	868.97%	51.84%	960.60%	230.07%
	LMM	754.39%	84.35%	730.18%	149.61%
	SABR	797.92%	143.15%	626.19%	172.31%
Long-term *IRS* (10-year)	**BDT**	614.95%	157.33%	567.85%	206.81%
	HW	663.89%	105.24%	570.13%	201.70%
	BK	640.23%	139.78%	603.49%	185.72%
	HJM	648.41%	121.51%	579.88%	205.46%
	G2++	770.59%	74.30%	593.49%	212.27%
	LMM	683.66%	98.93%	527.89%	148.89%
	SABR	624.00%	213.85%	560.77%	210.23%

Table 6: Difference in Credit Valuation Adjustment Calculations Between Dependency Models (Crisis Period)

This table displays the difference in counter party credit risk (CCR) calculations for different dependency models (independent, Gaussian, maximum), when measured by the credit valuation adjustment (CVA) of long–short positions in interest rate swaps (IRS) of different maturities (2, 5, and 10 years). The CVA is calculated only for the crisis period as the grey area in Figure 7, between 10 May 2005 to 1 August 2013. Six different interest rate models were used for the calculations: Black, Derman and Toy (BDT), Hull and White (HW), Black and Karasinski (BK), Heath,Jarrow, and Morton (HJM), Two-Factor Gaussian (G2++), and SABR models. Rates are in percentages (%). The interest rates for calculations were extracted from Bloomberg.

		Counter Party Credit Risk			
		Long (*IRS* Fixed-rate Payer)		**Short (*IRS* Float-rate Payer)**	
		CVA^{B_L} **vs.** $CVA_G^{B_L}$	$CVA_G^{B_L}$ **vs.** $CVA_{WWR}^{B_L}$	CVA^{B_S} **vs.** $CVA_G^{B_S}$	$CVA_G^{B_S}$ **vs.** $CVA_{WWR}^{B_S}$
Short-term *IRS* (2-years)	**BDT**	1164.47%	45.94%	739.82%	129.88%
	HW	1223.37%	36.81%	724.33%	130.62%
	BK	1381.87%	25.28%	777.88%	108.94%
	HJM	1184.20%	48.07%	734.54%	133.47%
	G2++	873.78%	43.13%	1379.50%	405.89%
	LMM	832.10%	77.81%	1027.13%	186.83%
	SABR	1120.03%	63.46%	720.18%	132.83%
Mid-term *IRS* (5-year)	**BDT**	759.48%	65.34%	627.18%	155.22%
	HW	802.56%	47.77%	624.88%	153.28%
	BK	828.47%	23.87%	675.38%	133.54%
	HJM	777.33%	48.65%	628.71%	156.50%
	G2++	718.02%	26.01%	891.20%	261.97%
	LMM	710.35%	65.32%	730.94%	158.78%
	SABR	743.85%	119.02%	604.74%	161.76%
Long-term *IRS* (10-year)	**BDT**	584.43%	121.14%	556.11%	194.27%
	HW	628.35%	85.44%	561.68%	188.69%
	BK	611.52%	75.97%	603.17%	169.00%
	HJM	611.85%	86.29%	565.34%	191.52%
	G2++	637.93%	37.63%	560.58%	226.42%
	LMM	643.79%	79.33%	529.32%	157.01%
	SABR	598.79%	182.56%	543.51%	198.44%

Table 7: Difference in Credit Valuation Adjustment Calculations Between Dependency Models (Non-Crisis Period)

This table displays the difference in counter party credit risk (CCR) calculations for different dependency models (independent, Gaussian, maximum), when measured by the credit valuation adjustment (CVA) of long–short positions in interest rate swaps (IRS) of different maturities (2, 5, and 10 years). The CVA is calculated only for the crisis period as the non-grey area in Figure 7, between 10 May 2005 to 1 August 2013. Six different interest rate models were used for the calculations: Black, Derman and Toy (BDT), Hull and White (HW), Black and Karasinski (BK), Heath, Jarrow, and Morton (HJM), Two-Factor Gaussian (G2++), and SABR models. Rates are in percentages (%). The interest rates for calculations were extracted from Bloomberg.

		Counter Party Credit Risk			
		Long (***IRS*** Fixed-rate Payer)		Short (***IRS*** Float-rate Payer)	
		CVA^{B_L} **vs.** $CVA^{B_L}_G$	$CVA^{B_L}_G$ **vs.** $CVA^{B_L}_{WWR}$	CVA^{B_S} **vs.** $CVA^{B_S}_G$	$CVA^{B_S}_G$ **vs.** $CVA^{B_S}_{WWR}$
Short-term *IRS* (2-years)	**BDT**	1439.14%	136.85%	745.73%	151.49%
	HW	1481.46%	79.66%	727.57%	149.73%
	BK	1772.48%	161.26%	767.05%	133.25%
	HJM	1438.69%	128.49%	741.67%	156.01%
	G2++	1165.57%	73.04%	1622.22%	344.29%
	LMM	900.03%	107.78%	1044.23%	180.96%
	SABR	1395.00%	103.63%	745.59%	152.47%
Mid-term *IRS* (5-year)	**BDT**	861.17%	126.90%	645.60%	174.19%
	HW	904.53%	81.13%	637.28%	171.68%
	BK	934.56%	128.66%	676.65%	156.24%
	HJM	890.40%	111.76%	651.03%	176.65%
	G2++	985.14%	66.82%	991.05%	217.36%
	LMM	778.05%	93.78%	729.85%	145.69%
	SABR	837.24%	158.95%	637.48%	177.61%
Long-term *IRS* (10-year)	**BDT**	634.36%	178.19%	573.81%	212.78%
	HW	685.75%	116.25%	574.40%	207.86%
	BK	656.56%	173.24%	603.75%	193.40%
	HJM	672.72%	142.54%	587.31%	212.15%
	G2++	871.64%	95.31%	607.54%	206.63%
	LMM	705.16%	108.54%	527.31%	145.40%
	SABR	640.97%	232.94%	569.88%	216.01%